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Decomposing locally compact groups into simple pieces

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Abstract

We present a contribution to the structure theory of locally compact groups. The emphasis is on compactly generated locally compact groups which admit no infinite discrete quotient. It is shown that such a group possesses a characteristic cocompact subgroup which is either connected or admits a non-compact non-discrete topologically simple quotient. We also provide a description of characteristically simple groups and of groups all of whose proper quotients are compact. We show that Noetherian locally compact groups without infinite discrete quotient admit a subnormal series with all subquotients compact, compactly generated Abelian, or compactly generated topologically simple.

Two appendices introduce results and examples around the concept of *quasi-product*.

1. *Introduction*

On structure theory. The structure of *finite* groups can to a large extent be reduced to finite simple groups and the latter have famously been classified (see e.g. [**GLS94**] *sqq.*).

For general locally compact groups, both the reduction to simple groups and the study of the latter still constitute major challenges. The connected case has found a satisfactory answer: indeed, the solution to Hilbert's fifth problem (see [**MZ55**, 4·6]) reduces the question to Lie theory upon discarding compact kernels. Lie groups are then described by investigating separately the soluble groups and the simple factors, which have been classified since the time of É. Cartan.

The contemporary structure problem therefore regards totally disconnected groups; there is not yet even a conjectural picture of a structure theory. In fact, until recently, the only structure theorem on totally disconnected locally compact groups was this single sentence in van Dantzig's 1931 thesis:

> "*Een (gesloten) Cantorsche groep bevat willekeurig kleine open ondergroepen.*" ([**vD31**, III Section 1, TG 38, p. 18])

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Recent progress, including statements on simple groups, is provided by the work of G. Willis [**Wil94**, **Wil07**]. New examples of simple groups have appeared in the geometric context of trees and of general buildings.

As results and examples for simple groups are being discovered, it becomes desirable to have a reduction step to simple groups – in parallel to the known cases of finite and connected groups. However, any reasonable attempt at classification must in one way or another exclude *discrete* groups: the latter are widely considered to be unclassifiable; this opinion can be given a mathematical content as e.g. in [**TV99**]. The discrete case also illustrates that there may be no simple (infinite) quotient, or even subquotient, at all.

The following trichotomy shows that, away from the unavoidable discrete situation, there is a compelling first answer.

THEOREM A. *Let G be a compactly generated locally compact group. Then exactly one of the following holds*:

(i) *G has an infinite discrete quotient*;
(ii) *G has a cocompact normal subgroup that is connected and soluble*;
(iii) *G has a cocompact normal subgroup that admits exactly n non-compact simple quotients (and no non-trivial discrete quotient), where* $0 < n < \infty$.

By a *simple* group, we mean a *topologically simple* group, i.e. a group all of whose Hausdorff quotients are trivial. Since a cocompact closed subgroup of a compactly generated locally compact group is itself compactly generated [**MŚ59**], it follows that the n simple quotients appearing in (iii) of Theorem A are compactly generated. (We always implicitly endow quotient groups with the quotient topology.)

The above theorem describes the *upper structure* of G. The first alternative can be made more precise in combination with the well-known (and easy to establish) fact that an infinite finitely generated group either admits an infinite residually finite quotient or has a finite index subgroup which admits an infinite simple quotient. In some sense, Theorem A plays the rôle of a non-discrete analogue of the latter fact; notice however that the finiteness of the number n of simple sub-quotients in case (iii) above is particular to non-discrete groups. It turns out that the above theorem is supplemented by the following description of the *lower structure* of G, which does not seem to have any analogue in the discrete case.

THEOREM B. *Let G be a compactly generated locally compact group. Then one of the following holds*:

(i) *G has an infinite discrete normal subgroup*;
(ii) *G has a non-trivial closed normal subgroup which is compact-by-{connected soluble}*;
(iii) *G has exactly n non-trivial minimal closed normal subgroups, where* $0 < n < \infty$.

The normal subgroups appearing in the above have no reason to be compactly generated in general. On another hand, since any (Hausdorff) quotient of a compactly generated group is itself compactly generated, it follows that Theorem B may be applied repeatedly to the successive quotients that it provides. Such a process will of course not terminate after finitely many steps in general. However, if G satisfies additional finiteness conditions, this recursive process can indeed reach an end in finite time. In order to make this precise, we introduce the following terminology. We call a topological group G **Noetherian** if it satisfies the ascending chain condition on open subgroups. Obvious examples are provided by compact (e.g. finite) groups, connected groups and polycyclic groups. If G is locally compact, then

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G is Noetherian if and only if every open subgroup is compactly generated. (Warning: the notion introduced in [**Gui73**, Section III] is more restrictive as it posits compact generation of all *closed* subgroups.)

THEOREM C. *Let G be a locally compact Noetherian group. Then G possesses an open normal subgroup G_k and a finite series of closed subnormal subgroups*

$$1 = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_k \lhd G$$

such that, for each $i \leqslant k$, the subquotient G_i/G_{i-1} is either compact, or isomorphic to $\mathbf{Z}$ *or* $\mathbf{R}$*, or topologically simple non-discrete and compactly generated.*

In the special case of connected groups, the existence of the above decomposition follows easily from the solution to Hilbert's fifth problem. In that case, the simple subquotients are connected non-compact adjoint simple Lie groups, while the presence of discrete free Abelian groups accounts for possible central extensions of simple Lie groups such as $\widetilde{\mathrm{SL}_2(\mathbf{R})}$ (an example going back to Schreier [**Sch25**, Section 5 Beispiel 2]). No analogue of Theorem C seems to be known for *discrete* Noetherian groups.

Characteristically simple groups and quasi-products. By a **characteristic subgroup** of a topological group G, we mean a closed subgroup which is preserved by every topological group automorphism of G. A group admitting no non-trivial such subgroup is called **characteristically simple**. This property is satisfied by any *minimal* normal subgroup, for instance those occurring in Theorem B.

In fact, our above results lead also to a description of characteristically simple groups, as follows.

COROLLARY D. *Let G be a compactly generated locally compact group. If G is characteristically simple, then one of the following holds*:

(i) *G is discrete*;
(ii) *G is compact*;
(iii) *$G \cong \mathbf{R}^n$ for some n*;
(iv) *G is a quasi-product with topologically simple pairwise isomorphic quasi-factors.*

By definition, a topological group G is called a **quasi-product** with **quasi-factors** $N_1, \dots, N_p$ if N_i are closed normal subgroups such that the multiplication map

$$N_1 \times \cdots \times N_p \longrightarrow G$$

is injective with dense image. Usual direct products are obvious examples, but the situation is much more complicated for general totally disconnected groups. The above definition degenerates in the commutative case; for instance, $\mathbf{R}$ is a quasi-product with quasi-factors $\mathbf{Z}$ and $\sqrt{2}\mathbf{Z}$. Several *centrefree* examples of quasi-products, including characteristically simple ones, will be constructed in Appendix II. However, as of today we are not aware of any *compactly generated* characteristically simple group which falls into Case (iv) of Corollary D without being a genuine direct product. As we shall see in Appendix II, the existence of such an example implies the existence of a compactly generated topologically simple locally compact group admitting a proper dense normal subgroup.

Groups with every proper quotient compact. The first goal that we shall pursue in this paper is to describe the compactly generated locally compact groups which admit only compact proper quotients. The non-compact groups satisfying this condition are sometimes

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called **just-non-compact**. In the discrete case, the corresponding notion is that of **just-infinite groups**, namely discrete groups all of whose proper quotients are finite. A description of these was given by J. S. Wilson in a classical article ([**Wil71**, Proposition 1]). Anticipating on the terminology introduced below, we can epitomise our contribution to this question as follows:

A just-non-compact group is either discrete or monolithic.

The most obvious case where a topological group has only compact quotients is when it is **quasi-simple**, which means that it possesses a cocompact normal subgroup which is topologically simple and contained in every non-trivial closed normal subgroup.

This situation extends readily to the following. We say that a topological group is **monolithic** with **monolith** L if the intersection of all non-trivial closed normal subgroups is itself a *non-trivial* group L. Non-quasi-simple examples are provided by the standard wreath product of a topologically simple group by a finite transitive permutation group (see [**Wil71**, Construction 1]). Notice that the monolith is necessarily characteristically simple.

In the discrete case, groups with only finite proper quotients can be very far from monolithic, indeed residually finite: examples are provided by all lattices in connected centrefree simple Lie groups of rank at least two in view of a fundamental theorem of G. Margulis [**Mar91**]; for instance, $\mathbf{PSL}_3(\mathbf{Z})$ (this particular case was already known to J. Mennicke [**Men65**]). The following result shows that such examples do not exist in the non-discrete case.

THEOREM E. *Let G be a compactly generated non-compact locally compact group such that every non-trivial closed normal subgroup is cocompact. Then one of the following holds*:

(i) *G is monolithic and its monolith is a quasi-product with finitely many isomorphic topologically simple groups as quasi-factors*;

(ii) *G is monolithic with monolith $L \cong \mathbf{R}^n$. Moreover G/L is isomorphic to a closed irreducible subgroup of $\mathbf{O}(n)$. In particular G is an almost connected Lie group*;

(iii) *G is discrete and residually finite.*

We shall see in Section 3 below that this theorem yields a topological simplicity corollary for locally compact groups endowed with a BN-pair which supplements classical results by J. Tits [**Tit64**].

The proof of Theorem E relies on an analysis of filtering families of closed normal subgroups in totally disconnected locally compact groups, which is carried out in Proposition 2·5 below. As a by-product, it yields in particular a characterisation of residually discrete groups (see Corollary 4·1) and the following easier companion to Theorem E, where $\mathrm{Res}(G)$ denotes the **discrete residual** of G, namely the intersection of all open normal subgroups.

THEOREM F. *Let G be a compactly generated locally compact group all of whose discrete quotients are finite. Then* $\mathrm{Res}(G)$ *is a cocompact characteristic closed subgroup of G without non-trivial discrete quotient.*

The cocompact subgroup $\mathrm{Res}(G)$ is compactly generated (see [**MŚ59**]). Since any compact totally disconnected group is a profinite group and, hence, admits numerous discrete quotients, it follows that, under the hypotheses of Theorem F, *any compact quotient of the discrete residual is connected*. Loosely speaking, the discrete residual is thus a sort of *cocompact core* of the group G. As a consequence, we obtain the following.

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COROLLARY G. *Let G be a compactly generated totally disconnected locally compact group all of whose discrete quotients are finite. Then the discrete residual of G is cocompact and admits no non-trivial discrete or compact quotient.*

Theorem A follows easily by combining Theorem E with the fact, due to R. Grigorchuk and G. Willis, that any non-compact compactly generated locally compact group admits a just-non-compact quotient (unpublished, see Proposition 5·2 below).

2. *Basic tools*

Generalities on locally compact groups. In this preliminary section, we collect a number of subsidiary facts on locally compact groups which will be used repeatedly in the sequel. Unless specified otherwise, all topological groups are assumed Hausdorff.

We will frequently invoke the following well-known statement without explicit reference.

LEMMA 2·1. *If a closed subgroup of a compactly generated locally compact group is cocompact, then it is itself compactly generated.*

Proof. See [**MŚ59**, Corollary 2]. □

We say that a subgroup H of a topological group G is **topologically locally finite** if every finite subset of H is contained in a compact subgroup of G. Any locally compact group G possesses a maximal normal topologically locally finite subgroup which is closed and called the **LF-radical** and denoted $\mathrm{Rad}_{\mathscr{LF}}(G)$; another important fact is that any compact subset of a locally compact topologically locally finite group is contained in a compact subgroup. We refer to [**Pla65**] and [**Cap09**, Section 2] for details.

It is well known that the LF-radical is compact for connected groups:

LEMMA 2·2. *Every connected locally compact group admits a maximal compact normal subgroup. Moreover, the corresponding quotient is a connected Lie group.*

Proof. The solution to Hilbert's fifth problem [**MZ55**, Theorem 4·6] provides a compact normal subgroup such that the quotient is a Lie group; now the statement follows from the corresponding fact for connected Lie groups. □

As a further element of terminology, the **quasi-centre** of a topological group G is the subset $\mathscr{QZ}(G)$ consisting of all those elements possessing an open centraliser. The subgroup $\mathscr{QZ}(G)$ is topologically characteristic in G, but need not be closed. Since any element with a discrete conjugacy class possesses an open centraliser, it follows that the quasi-centre contains all discrete normal subgroups of G.

We shall use the following result of Ušakov, for which we recall that a topological group is called **topologically $\overline{\mathbf{FC}}$** if every conjugacy class has compact closure.

THEOREM 2·3 (Ušakov [**Uša63**]). *Let G be a locally compact topologically $\overline{FC}$-group. Then the union of all compact subgroups of G forms a closed normal subgroup, which therefore coincides with $\mathrm{Rad}_{\mathscr{LF}}(G)$, and the corresponding quotient $G/\mathrm{Rad}_{\mathscr{LF}}(G)$ is Abelian.*

Moreover, if in addition G is compactly generated, then G is compact-by-Abelian. More precisely, $\mathrm{Rad}_{\mathscr{LF}}(G)$ is compact and $G/\mathrm{Rad}_{\mathscr{LF}}(G)$ is isomorphic to $\mathbf{R}^n \times \mathbf{Z}^m$ for some $n, m \geqslant 0$.

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The following consideration provides a necessary (and sufficient) condition for the group considered in Theorem E to admit a non-trivial *discrete* normal subgroup.

PROPOSITION 2·4. *Let G be a compactly generated non-compact locally compact group such that every non-trivial closed normal subgroup is cocompact. If G admits a non-trivial discrete normal subgroup, then G is either discrete or $\mathbf{R}^n$-by-finite.*

Proof. Let $H \lhd G$ be a non-trivial discrete normal subgroup. Then H is cocompact and, hence, it is a cocompact lattice in the compactly generated group G. Therefore H is finitely generated. We deduce that the normal subgroup $\mathscr{Z}_G(H) \lhd G$ is open, since every element of H has a discrete conjugacy class and, hence, an open centraliser. In particular, G is discrete if $\mathscr{Z}_G(H)$ is trivial and we can therefore assume that $\mathscr{Z}_G(H)$ is cocompact.

The quotient $\mathscr{Z}_G(H)/\mathscr{Z}(H)$ is compact since it sits as open (hence closed) subgroup in G/H. Hence, the centre of $\mathscr{Z}_G(H)$ is cocompact for it contains $\mathscr{Z}(H)$. Thus Theorem 2·3 applies to $\mathscr{Z}_G(H)$. We claim that the LF-radical of $\mathscr{Z}_G(H)$ is trivial: otherwise, being normal in G, it would be cocompact, hence compactly generated, thus compact ([**Cap09**, Lemma 2·3]) and now finally trivial after all since G is non-compact by hypothesis. In conclusion, $\mathscr{Z}_G(H)$ is isomorphic to $\mathbf{R}^n \times \mathbf{Z}^m$. In addition, n or m vanishes since the identity component of $\mathscr{Z}_G(H)$ is normal in G and hence trivial or cocompact. We conclude by recalling that $\mathscr{Z}_G(H)$ is open and cocompact, thus of finite index in G. □

Filters of closed normal subgroups. We continue with another general fact on compactly generated groups, which was the starting point of this work. The argument was inspired by a reading of [**BM00**, Lemma 1·4·1]; it also plays a key rôle in the structure theory of isometry groups of non-positively curved spaces developed in [**CM09**].

PROPOSITION 2·5. *Let G be a compactly generated totally disconnected locally compact group. Then any identity neighbourhood V contains a compact normal subgroup Q_V such that any filtering family of non-discrete closed normal subgroups of the quotient G/Q_V has non-trivial intersection.*

Proof. Let $\mathfrak{g}$ be a **Schreier graph** for G associated to a compact open subgroup $U < G$ contained in the given identity neighbourhood V (which exists by a classical result in D. van Dantzig's 1931 thesis [**vD31**, III Section 1, TG 38, p. 18], see [**Bou71**, III Section 4, no. 6]). We recall that $\mathfrak{g}$ is obtained by defining the vertex set as G/U and drawing edges according to a compact generating set which is a union of double cosets modulo U; see [**Mon01**, Section 11·3]. The kernel of the G-action on $\mathfrak{g}$ is nothing but $Q_V = \bigcap_{g \in G} gUg^{-1}$ which is compact and contained in V.

Since we are interested in closed normal subgroups of the quotient G/Q_V, there is no loss of generality in assuming Q_V trivial. In other words, we assume henceforth that G acts faithfully on $\mathfrak{g}$. Let v_0 be a vertex of $\mathfrak{g}$ and denote by $v_0^\perp$ the set of neighbouring vertices. Since G is vertex-transitive on $\mathfrak{g}$, it follows that for any normal subgroup $N \lhd G$, the N_{v_0}-action on $v_0^\perp$ defines a finite permutation group $F_N < \mathrm{Sym}(v_0^\perp)$ which, as an abstract permutation group, is independent of the choice of v_0. Therefore, if N is non-discrete, this permutation group F_N has to be non-trivial since U is open and $\mathfrak{g}$ connected. Now a filtering family $\mathscr{F}$ of non-discrete normal subgroups yields a filtering family of non-trivial finite subgroups of $\mathrm{Sym}(v_0^\perp)$. Thus the intersection of these finite groups is non-trivial. Let g be a non-trivial element in this intersection. For any $N \in \mathscr{F}$, let N_g be the inverse image of $\{g\}$ in N_{v_0}. Thus N_g is a non-empty compact subset of N for each $N \in \mathscr{F}$. Since the family $\mathscr{F}$ is filtering, so are $\{N_{v_0} \mid N \in \mathscr{F}\}$ and $\{N_g \mid N \in \mathscr{F}\}$. The result follows,

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since a filtering family of non-empty closed subsets of the compact set G_{v_0} has a non-empty intersection. $\square$

Minimal normal subgroups. With Proposition 2·5 at hand, we deduce the following. An analogous result for the upper structure of totally disconnected groups will be established in Section 5 below (see Proposition 5·4).

PROPOSITION 2·6. *Let G be a compactly generated totally disconnected locally compact group which possesses no non-trivial compact or discrete normal subgroup. Then every non-trivial closed normal subgroup of G contains a minimal one, and the set $\mathscr{M}$ of non-trivial minimal closed normal subgroups is finite. Furthermore, for any proper subset $\mathscr{E} \subset \mathscr{M}$, the subgroup $\overline{\langle M \mid M \in \mathscr{E} \rangle}$ is properly contained in G.*

Proof. In view of Proposition 2·5, any chain of non-trivial closed normal subgroups of G has a minimal element. Thus Zorn's lemma ensures that the set $\mathscr{M}$ of minimal non-trivial closed normal subgroups of G is non-empty, and that any non-trivial closed normal subgroup contains an element of $\mathscr{M}$.

In order to establish that $\mathscr{M}$ is finite, we use the following notation. For each subset $\mathscr{E} \subseteq \mathscr{M}$ we set $M_{\mathscr{E}} = \langle M \mid M \in \mathscr{E} \rangle$; $\overline{M}_{\mathscr{E}}$ denotes its closure. The following argument was inspired by the proof of [**BM00**, Proposition 1·5·1].

We claim that if $\mathscr{E}$ is a proper subset of $\mathscr{M}$, then $\overline{M}_{\mathscr{E}}$ is a proper subgroup of G. Indeed, for all $M \in \mathscr{E}$ and $M' \in \mathscr{M} \setminus \mathscr{E}$ we have $[M, M'] \subseteq M \bigcap M' = 1$. Thus M' centralises $\overline{M}_{\mathscr{E}}$. In particular, if $\overline{M}_{\mathscr{E}} = G$, then M' centralises G. Thus $M' \leqslant \mathscr{Z}(G)$ is Abelian, and any proper subgroup of M' is normal in G. Since M' is a minimal normal subgroup, it follows that M' has no proper closed subgroup. The only totally disconnected locally compact groups with this property being the cyclic groups of prime order, we deduce that M' is finite, which contradicts the hypotheses. The claim stands proven.

Consider now the family

$$\mathscr{N} = \{\overline{M}_{\mathscr{M} \setminus \mathscr{F}} \mid \mathscr{F} \subseteq \mathscr{M} \text{ is finite}\}$$

of closed normal subgroups of G. This family is filtering. Furthermore the above claim shows that $\overline{M}_{\mathscr{M} \setminus \mathscr{F}}$ is properly contained in G whenever $\mathscr{F}$ is non-empty. Since $\bigcap \mathscr{N} = 1$, it follows from Proposition 2·5 that $\mathscr{N}$ is finite, and hence $\mathscr{M}$ is so, as desired. $\square$

Just-non-compact groups.

Proof of Theorem E. We shall use repeatedly the fact that every normal subgroup of G has trivial LF-radical, which is established as in the above proof of Proposition 2·4. In particular, normal subgroups of G have no non-trivial compact normal subgroups.

We begin by treating the case where G is totally disconnected. Let L be the intersection of all non-trivial closed normal subgroups. We distinguish two cases.

If L is trivial, then Proposition 2·5 shows that G admits a non-trivial discrete normal subgroup. Thus Proposition 2·4 applies and G is discrete; in that case, Wilson's result ([**Wil71**, Proposition 1]) completes the proof.

Assume now that L is not trivial. Then it is cocompact whence compactly generated since G is so. Notice that by definition L is characteristically simple. We further distinguish two cases.

On the one hand, assume that the quasi-centre $\mathscr{QZ}(L)$ is non-trivial. Then it is dense in L. Since L is compactly generated, the arguments of the proof of [**BEW08**, Theorem 4·8]

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(see Proposition 4·3 below) show that L possesses a compact open normal subgroup. Since L has no non-trivial compact normal subgroup, we deduce that L is discrete. Now L is a non-trivial discrete normal subgroup and we have already seen above how to finish the proof in that situation.

On the other hand, assume that the quasi-centre $\mathscr{QZ}(L)$ is trivial. In particular L possesses no non-trivial discrete normal subgroup and we deduce from Proposition 2·6 that the set $\mathscr{M}$ of non-trivial minimal closed normal subgroups of L is finite and non-empty. Since L has no non-trivial compact normal subgroup, no element of $\mathscr{M}$ is compact.

The group G acts on $\mathscr{M}$ by conjugation. Let $\mathscr{E}$ denote a G-orbit in $\mathscr{M}$. Since $M_{\mathscr{E}} = \langle M \mid M \in \mathscr{E} \rangle$ is normal in G, it is dense in L. By Proposition 2·6, we infer that $\mathscr{E} = \mathscr{M}$. In other words G acts transitively on $\mathscr{M}$.

It now follows that L is a quasi-product with the elements of $\mathscr{M}$ as quasi-factors. In particular, any normal subgroup M' of any $M \in \mathscr{M}$ is normalised by M and centralised by each $N \in \mathscr{M}$ different from M. Since $\prod_{N \in \mathscr{M}} N$ is dense in L, we infer that M' is normal in L. Since M is a minimal normal subgroup of L, it follows that M is topologically simple and (i) holds.

Now we turn to the case where G is not totally disconnected, hence its identity component G° is cocompact. Since the maximal compact normal subgroup of Lemma 2·2 is trivial, G° is a connected Lie group. Since its soluble radical is characteristic, it is trivial or cocompact.

In the former case, G° is semi-simple without compact factors. Since its isotypic factors are characteristic, there is only one isotypic factor and we conclude that (i) holds.

If on the other hand the radical of G° is cocompact, we deduce that G admits a characteristic cocompact connected soluble subgroup $R \lhd G$. Let T be the last non-trivial term of the derived series of R. If the identity component T° is trivial, then T is a non-trivial discrete normal subgroup of G and we may conclude by means of Proposition 2·4. Otherwise, the group R possesses a characteristic connected Abelian subgroup T°, which is thus cocompact in G.

Since T° has no non-trivial compact subgroup, we have $T^\circ \cong \mathbf{R}^d$ for some d. The kernel of the homomorphism $G \to \mathrm{Out}(T^\circ) = \mathrm{Aut}(T^\circ)$ is a cocompact normal subgroup N containing T° in its centre, and such that N/T° is compact. In particular N is a compactly generated locally compact group in which every conjugacy class is relatively compact. In view of Ušakov's result (Theorem 2·3) and of the triviality of $\mathrm{Rad}_{\mathscr{LF}}(N)$, the group N is of the form $\mathbf{R}^n \times \mathbf{Z}^m$. In conclusion, since T° is cocompact in N, we have $T^\circ = N \cong \mathbf{R}^n$. Considering once again the map $G \to \mathrm{Aut}(T^\circ) \cong \mathbf{GL}_n(\mathbf{R})$, we deduce that G/T° is isomorphic to a compact subgroup of $\mathbf{GL}_n(\mathbf{R})$, which has to be irreducible since otherwise T° would contain a non-cocompact subgroup normalised by G. We conclude the proof of Theorem E by recalling that every compact subgroup of $\mathbf{GL}_n(\mathbf{R})$ is conjugated to a subgroup of $\mathbf{O}(n)$. □

3. *Topological BN-pairs*

By a celebrated lemma of Tits [**Tit64**], any group admitting a BN-pair of irreducible type has the property that a normal subgroup acts either trivially or chamber-transitively on the associated building. Tits used his transitivity lemma to show in *loc. cit.* that if G is perfect and possesses a BN-pair with B soluble, then any non-trivial normal subgroup is contained in $Z = \bigcap_{g \in G} gBg^{-1}$. More generally, the same conclusion holds provided G is perfect and B possesses a soluble normal subgroup U whose conjugates generate G. If G is endowed with a group topology, the same arguments show that if G is topologically perfect and U

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is pro-soluble, then G/Z is topologically simple. The following consequence of Theorem E does not require any assumption on the normal subgroup structure of B.

COROLLARY 3·1. *Let G be a topological group endowed with a BN-pair of irreducible type, such that $B < G$ is compact and open. Then G has a closed cocompact (topologically) characteristic subgroup H containing $Z = \bigcap_{g \in G} gBg^{-1}$, such that H/Z is topologically simple.*

It follows in particular that G is *topologically commensurable* to a topologically simple group since Z is compact and H cocompact.

Proof. The assumption that B is compact open implies that G is locally compact and that the building X associated with the given BN-pair is locally finite. Since the kernel of this action coincides with $Z = \bigcap_{g \in G} gBg^{-1}$, we may and shall assume that G acts faithfully on X. Since G acts chamber-transitively on X and since B is the stabiliser of some base chamber c_0, it follows that G is generated by the union of B with a finite set of elements mapping c_0 to each of its neighbours. Thus G is compactly generated. By Tits' transitivity lemma [**Tit64**, Proposition 2·5], for any non-trivial normal subgroup N of G, we have $G = N.B$, whence N is cocompact provided it is closed.

If X is finite, then G is compact and we are done. Otherwise G is non-compact and non-discrete, because B is then necessarily infinite. We are thus in a position to apply Theorem E. Since G is a subgroup of the totally disconnected group $\mathrm{Aut}(X)$, it follows that G is totally disconnected and we deduce that G is monolithic with a quasi-product of topologically simple groups as a monolith. The fact that the monolith has only one simple factor follows from the fact that G acts faithfully, minimally and without fixed point at infinity on a CAT(0) realisation of X. Such a CAT(0) realisation is necessarily irreducible as a CAT(0) space by [**CH06**], and [**CM09**, Theorem 1·10] ensures that no abstract normal subgroup of G splits non-trivially as a direct product. □

4. *Discrete quotients*

Residually discrete groups. Any topological group admits a filtering family of closed normal subgroups, consisting of all *open* normal subgroups. Specialising Proposition 2·5 to this family yields the following fact.

COROLLARY 4·1. *Let G be a compactly generated locally compact group. Then the following assertions are equivalent*:

(i) *G is residually discrete*;

(ii) *G is a totally disconnected SIN-group*;

(iii) *The compact open normal subgroups form a basis of identity neighbourhoods.*

Recall that a locally compact group is called a **SIN-group** if it possesses a basis of identity neighbourhoods which are invariant by conjugation. Equivalently, SIN-groups are those for which the left and right uniform structures coincide. A classical theorem of Freudenthal and Weil ([**Fre36**] and [**Wei40**, Section 32]; see also [**Dix96**, Section 16·4·6]) states that a connected group is SIN if and only if it is of the form $K \times \mathbf{R}^n$ with K compact (connected) and $n \geqslant 0$. This complements nicely the above corollary, implying easily that *any* locally compact SIN-group is an extension of a discrete group by a group $K \times \mathbf{R}^n$; the latter consequence is [**GM71**, Theorem 2·13(1)].

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Proof of Corollary 4·1. The implications (iii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii) are immediate.

(ii) $\Rightarrow$ (iii) Since G is totally disconnected, the compact open subgroups form a basis of identity neighbourhoods [**Bou71**, III Section 4, No. 6]. By assumption, given any compact open subgroup $U < G$, there is an identity neighbourhood $V \subseteq U$ which is invariant by conjugation. The subgroup generated by V is thus normal in G, open since it contains V and compact since it is contained in U. Thus (iii) holds indeed.

(i) $\Rightarrow$ (iii) Assume that G is residually discrete. Then G is totally disconnected and the compact open subgroups form a basis of identity neighbourhoods. Let $V < G$ be compact and open and $Q_V \subseteq V$ denote the compact normal subgroup provided by Proposition 2·5. By assumption we have $\bigcap \mathscr{F} = 1$, where $\mathscr{F}$ denotes the collection of all open normal subgroups of G.

We claim that the quotient G/Q_V is residually discrete. Indeed, for any $x \in G$, the family $\{(x \cdot N) \cap Q_V\}_{N \in \mathscr{F}}$ is filtering and its intersection coincides with $\{x\} \cap Q_V$. Since Q_V is compact and since open subgroups are closed, it follows that for each $x \notin Q_V$ there exist finitely many elements $N_1, \ldots, N_k \in \mathscr{F}$ such that $(\bigcap_{i=1}^k x \cdot N_i) \cap Q_V = \varnothing$. Since $N_0 = \bigcap_{i=1}^k N_i$ belongs to $\mathscr{F}$, we have thus found an open normal subgroup N_0 of G such that $x \notin N_0 \cdot Q_V$. The projection of $N_0 \cdot Q_V$ in the quotient G/Q_V is thus an open normal subgroup which avoids the projection of x. This proves the claim.

Since the collection of all open normal subgroups of G/Q_V forms a filtering family, Proposition 2·5 now implies that G/Q_V possesses some *discrete* open normal subgroup. Therefore G/Q_V is itself discrete and hence Q_V is open in G. This shows that any compact open subgroup V contains a compact open normal subgroup Q_V. The desired conclusion follows. $\square$

Remark 4·2. Notice that a profinite extension of a discrete group is not necessarily residually discrete, as illustrated by the unrestricted wreath product $(\prod_{\mathbf{Z}} \mathbf{Z}/2) \rtimes \mathbf{Z}$, where $\mathbf{Z}$ acts by shifting indices. However, if a totally disconnected group G possesses a compact open normal subgroup Q which is topologically finitely generated, then G is residually discrete. Indeed, the profinite group Q has then finitely many subgroups of any given finite index and, hence, possesses a basis of identity neighbourhoods consisting of *characteristic* subgroups.

The discrete residual. We recall that the **discrete residual** of a topological group is the intersection of all open normal subgroups. Notice that the quotient of a group by its discrete residual is residually discrete.

Proof of Theorem F. Let R_0 denote the discrete residual of G. Since G/R_0 is residually discrete, it follows from Corollary 4·1 that R_0 is contained cocompactly in some open normal subgroup of G. In view of the hypotheses, this shows that R_0 is cocompact, whence compactly generated.

Let R_1 denote the discrete residual of R_0. We have to show that $R_0 = R_1$. Since R_1 is characteristic in R_0, it is normal in G. Observe that all subquotients of G/R_1 considered below are totally disconnected since the latter is an extension of the residually discrete groups R_0/R_1 and G/R_0. We consider the canonical projection $\pi : G \to G/R_1$ and define an intermediate characteristic subgroup $R_1 \leqslant L \leqslant R_0$ by

$$L = \pi^{-1}(\mathrm{Rad}_{\mathscr{LF}}(R_0/R_1)).$$

Since the group R_0/R_1 is residually discrete, it follows from Corollary 4·1 that its LF-radical is open. In other words the subquotient R_0/L is discrete. Since it is cocompact in G/L, it is moreover finitely generated. It follows that the normal subgroup $Z := \mathscr{Z}_{G/L}(R_0/L)$ is

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open. By hypothesis, it has finite index in G/L and contains R_0/L. In particular, it has cocompact centre and thus Z is compact-by-$\mathbf{Z}^m$ for some m, as is checked e.g. as an easy case of Theorem 2·3, recalling that Z is totally disconnected. In conclusion, Z has a compact open characteristic subgroup; the latter has finite index in G/L by assumption. Thus L is cocompact in G, whence compactly generated. The topologically locally finite group L/R_1 is thus compact ([**Cap09**, Lemma 2·3]). In particular R_1 itself is cocompact in G. Now G/R_1 is a profinite group, thus residually discrete. This finally implies that $R_0 = R_1$, as desired. □

Quasi-discrete groups. We end this section with an additional remark regarding the quasi-centre. We shall say that a topological group is **quasi-discrete** if its quasi-centre is dense. Examples of quasi-discrete groups include discrete groups as well as profinite groups which are direct products of finite groups. A connected group is quasi-discrete if and only if it is Abelian. The following fact can be extracted from the proof of [**BEW08**, Theorem 4·8]; since the argument is short we include it for the sake of completeness.

PROPOSITION 4·3. *In any quasi-discrete compactly generated totally disconnected locally compact group, the compact open normal subgroups form a basis of identity neighbourhoods.*

Thus a compactly generated totally disconnected locally compact group which is quasi-discrete satisfies the equivalent conditions of Corollary 4·1.

Proof. Let G be as in the statement and $U < G$ be any compact open subgroup. By compact generation, there is a finite set $\{g_1, \dots, g_n\}$ that, together with U, generates G. Since G is quasi-discrete, $G = \mathscr{QZ}(G) \cdot U$ and thus we can assume that each g_i belongs to $\mathscr{QZ}(G)$. The subgroup $\bigcap_{i=1}^n \mathscr{Z}_U(g_i) < U$ is open and hence contains a finite index open subgroup V which is normalised by U. Thus V is a compact open normal subgroup of G contained in U. □

We finish this subsection by recording two consequences of the latter for the sake of future reference.

COROLLARY 4·4. *Let G be a compactly generated locally compact group without non-trivial compact normal subgroup. If G is quasi-discrete, then the identity component G° is open, central and isomorphic to $\mathbf{R}^n$ for some n. Moreover, we have $G = \mathscr{QZ}(G)$.*

Proof. We first observe that G° is central; indeed, it is centralised by the dense subgroup $\mathscr{QZ}(G)$ since G° is contained in every open subgroup. By Lemma 2·2, the LF-radical of G° is a compact normal subgroup of G, and is thus trivial by hypothesis; moreover, it follows that G° is an Abelian Lie group without periodic element. Thus $G^\circ \cong \mathbf{R}^n$ for some n.

Since any quotient of a quasi-discrete group remains quasi-discrete, Proposition 4·3 implies that the group of components G/G° admits some compact open normal subgroup V. It now suffices to prove $V = 1$ to establish the remaining statements.

Denote by $N \lhd G$ the G°-by-V extension; it is compactly generated and has only compact conjugacy classes since G° is central. In particular, Theorem 2·3 guarantees that $\mathrm{Rad}_{\mathscr{LF}}(N)$ is compact and that $N/\mathrm{Rad}_{\mathscr{LF}}(N)$ is Abelian without compact subgroup. Since N is normal in G, it follows that $\mathrm{Rad}_{\mathscr{LF}}(N)$ is trivial and thus indeed $N = G^\circ$ as required. □

COROLLARY 4·5. *Let G be a compactly generated totally disconnected locally compact group admitting an open quasi-discrete subgroup. Then either G is compact or G possesses an infinite discrete quotient.*

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Proof. Let H be the given quasi-discrete open subgroup of G. Let $h \in \mathscr{QZ}(H)$ be an element of the quasi-centre of H. Then $\mathscr{Z}_H(h)$ is open in H, from which we infer that $\mathscr{Z}_G(h)$ is open in G and hence $h \in \mathscr{QZ}(G)$. In particular, the closure $Z = \overline{\mathscr{QZ}(G)}$ is open in G.

If Z has infinite index, then G/Z is an infinite discrete quotient of G. Otherwise Z has finite index and is thus compactly generated. By Proposition 4·3, it follows that Z possesses a compact open normal subgroup. Since Z has finite index in G, we deduce that G itself possesses a compact open normal subgroup. The desired conclusion follows. □

5. *On structure theory*

Quasi-simple quotients. Before undertaking the proof of Theorems A and B, we record one additional consequence of Proposition 2·5. Before stating it, we recall from the introduction that a group is called **quasi-simple** if it possesses a cocompact normal subgroup which is topologically simple and contained in every non-trivial closed normal subgroup of G; in other words, a quasi-simple group is a monolithic group whose monolith is cocompact and topologically simple.

COROLLARY 5·1. *Let G be a compactly generated locally compact group and $\{N_v \mid v \in \Sigma\}$ be a collection of pairwise distinct closed normal subgroups of G such that for each $v \in \Sigma$, the quotient G/N_v is quasi-simple, non-discrete and non-compact. Suppose that $\bigcap_{v\in\Sigma} N_v = 1$. Then Σ is finite.*

Proof. We write $H_v := G/N_v$. By hypothesis each H_v is monolithic with simple cocompact monolith, which we denote by S_v. We claim that G has no non-trivial compact normal subgroup. Let indeed $Q \lhd G$ be a compact normal subgroup of G. By the assumptions made on H_v, the image of Q in H_v is trivial for each $v \in \Sigma$. Thus $Q \subseteq \bigcap_{v\in\Sigma} N_v$ and hence Q is trivial.

The same line of argument shows that G has no non-trivial soluble normal subgroup. In particular, the identity component G° is a connected semi-simple Lie group with trivial centre and no compact factor, see Lemma 2·2. Such a Lie group G° is the direct product of its simple factors. Moreover, G has an open characteristic subgroup of finite index which splits as a direct product of the form $G^\circ \times D$ determining some compactly generated totally disconnected group D, see e.g. (the proof of) [**Mon01**, Theorem 11·3·4] or [**BM02**, Section 3·4].

The identity component of each H_v coincides with the image of G° (since any quotient of a totally disconnected group is totally disconnected). Thus, whenever H_v is not totally disconnected, the hypothesis implies $S_v = H_v^\circ$ and H_v is a profinite extension of one of the simple factors of G°, and that each factor appears once.

At this point, we can and shall assume that G is totally disconnected.

In view of Proposition 2·4 and the assumption made on H_v, the group S_v is non-discrete for each $v \in \Sigma$. In particular, it follows that H_v has trivial quasi-centre.

Since the image of $\mathscr{QZ}(G)$ in H_v is contained in $\mathscr{QZ}(H_v)$, we deduce that $\mathscr{QZ}(G)N_v = N_v$ for all $v \in \Sigma$. In other words, we have $\mathscr{QZ}(G) < \bigcap_{v\in\Sigma} N_v = 1$ and we conclude that G has trivial quasi-centre.

Let now $\mathscr{F}$ be the filter of closed normal subgroups of G generated by $\{N_v \mid v \in \Sigma\}$. Since G has no compact non-trivial normal subgroup and no non-trivial discrete normal subgroup (as $\mathscr{QZ}(G) = 1$), we deduce from Proposition 2·5 that $\mathscr{F}$ is finite. Thus Σ is finite as well, as desired. □

Maximal normal subgroups. We shall need the following statement due to R. Grigorchuk and G. Willis; since it is unpublished, we provide a proof for the reader's convenience.

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PROPOSITION 5·2. *Let G be a totally disconnected compactly generated non-compact locally compact group. Then G admits a non-compact quotient with every proper quotient compact.*

Proof. By Zorn's lemma, it suffices to prove that for any chain $\mathscr{H}$ of non-cocompact closed normal subgroups $H \lhd G$, the group $M = \overline{\bigcup_{H \in \mathscr{H}} H}$ is still non-cocompact. If not, then M is compactly generated. Therefore, choosing a compact open subgroup $U < G$, the chain $\{H.U\}$ of groups is an open covering of M, whence there is $H \in \mathscr{H}$ with $H.U \supseteq M$. Then this H is cocompact, which is absurd. □

Remark 5·3. In view of the structure theory of connected groups [**MZ55**], the above Proposition holds also true in the non-totally-disconnected case.

The following is a dual companion to Proposition 2·6. Additional information in this direction will be provided in Proposition II·1 in Appendix II below.

PROPOSITION 5·4. *Let G be a compactly generated totally disconnected locally compact group which possesses no infinite discrete quotient, and let $H = \mathrm{Res}(G)$ be the discrete residual of G. Then every proper closed normal subgroup of H is contained in a maximal one, and the set $\mathscr{N}$ of proper maximal closed normal subgroups is finite.*

Proof. By Theorem F, the discrete residual H is cocompact in G, hence compactly generated. Furthermore it has no non-trivial finite quotient. Since H is totally disconnected, any compact quotient would be profinite, and we infer that H has no non-trivial compact quotient. Now the same argument as in the proof of Proposition 5·2 using Zorn's lemma shows that every proper closed normal subgroup of H is contained in a maximal one.

Let $\mathscr{N}$ denote the collection of all these. Any quotient H/N being topologically simple, hence quasi-simple, the finiteness of $\mathscr{N}$ follows readily from Corollary 5·1. □

Upper and lower structure.

Proof of Theorem A. We assume throughout that G is non-compact since otherwise (ii) holds trivially. Assume first that G is almost connected. In particular the neutral component G° coincides with the discrete residual of G. Let R denote the maximal connected soluble normal subgroup of G; this **soluble radical** is indeed well defined even if G is not a Lie group as proved by K. Iwasawa ([**Iwa49**, Theorem 15]; see also [**Pat88**, (3·7)]. If R is cocompact we are in case (ii) of the Theorem. Otherwise G is not amenable and using the structure theory of connected groups (notably [**MZ55**, Theorem 4·6]), we deduce that G°/R possesses a non-compact (Lie-)simple factor, so that all assertions of the case (iii) of the Theorem are satisfied.

We now assume that G is not almost connected. If G admits an infinite discrete quotient we are in case (i) of the Theorem. We assume henceforth that G has no infinite discrete quotient. In particular its discrete residual G^+ is cocompact and admits neither non-trivial discrete quotients nor disconnected compact quotients, see Corollary G. Moreover G^+ is compactly generated, non-compact and contains the identity component G°.

Let $\mathcal{S}$ be the collection of all topologically simple quotients of G^+. Applying Corollary 5·1 to the quotient group G^+/K, where $K = \bigcap_{S \in \mathcal{S}} \mathrm{Ker}(G^+ \to S)$, we deduce that $\mathcal{S}$ is finite. Thus the assertion (iii) of Theorem A will be established provided we show that $\mathcal{S}$ is non-empty.

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To this end, it suffices to prove that the group of components G^+/G° admits some non-compact topologically simple quotient. But this follows from Proposition 5·4 since any quotient of G^+/G° is non-compact and since each topologically simple quotient is afforded by a maximal closed normal subgroup. □

Proof of Theorem B. We assume that assertions (i) and (ii) of the Theorem fail. Note that if G is totally disconnected, then Proposition 2·6 finishes the proof.

As is well known (see the proof of Corollary 5·1), the non-existence of non-trivial compact (resp. connected soluble) normal subgroups implies that G possesses a characteristic open subgroup of finite index $G^+ < G$ which splits as a direct product of the form $G^+ = G^\circ \times D$, where G° is a semi-simple Lie group and D is totally disconnected.

Notice that G/G° is a totally disconnected locally compact group which might possess non-trivial finite normal subgroups. In order to remedy this situation, we shall now exhibit a closed normal subgroup $G_1 \leqslant G$ containing G° as a finite index subgroup and such that G/G_1 has no non-trivial compact or discrete normal subgroup.

Let N be a closed normal subgroup of G containing G°. Then $N^+ = N \cap G^+$ is a finite index subgroup which decomposes as a direct product of the form $N^+ \cong G^\circ \times (D \cap N)$. If the image of N in G/G° is compact (resp. discrete), then $D \cap N$ is a compact (resp. discrete) normal subgroup of G, and must therefore be trivial, since otherwise assertion (ii) (resp. (i)) would hold true. We deduce that any compact (resp. discrete) normal subgroup of G/G° is finite with order bounded above by $[G : G^+]$. In particular, there is a maximal such normal subgroup, and we denote by G_1 its pre-image in G.

Since $G_1 \cap D$ injects into G_1/G°, it is a finite normal subgroup of G and must therefore be trivial. Moreover $G^+ = G^\circ D$ is closed in G. Thus D has closed image in G/G°, whence in G/G_1 since the canonical projection $G/G^\circ \to G/G_1$ is proper, as it has finite kernel. This implies that DG_1 is closed in G. In other words $\langle D \cup G_1 \rangle$ is a characteristic closed subgroup of finite index in G which is isomorphic to $G_1 \times D$.

It follows at once that there is a canonical one-to-one correspondence between the closed normal subgroups of G contained in D and the closed normal subgroups of G/G_1 contained in DG_1/G_1.

Now G/G_1 is a compactly generated totally disconnected locally compact group without non-trivial compact or discrete normal subgroup, and Proposition 2·6 guarantees that the set $\mathscr{M}_1$ of its non-trivial minimal closed normal subgroups is finite and non-empty. Moreover, an element of $\mathscr{M}_1$ does not possess any non-trivial finite index closed normal subgroup and must therefore be contained in DG_1/G_1. Similarly, any minimal closed normal subgroup of G must be contained in G^+.

Since any minimal closed normal subgroup of G is either connected or totally disconnected, and since the connected ones are nothing but (regrouping of) simple factors of G°, we finally obtain a canonical one-to-one correspondence between $\mathscr{M}_1$ and the set of non-trivial minimal closed normal subgroups of G which are totally disconnected. The desired conclusion follows since, as observed above, the set $\mathscr{M}_1$ is finite and non-empty. □

Proof of Corollary D. Assume G is not discrete. The discrete residual $\mathrm{Res}(G)$ is characteristic. If $\mathrm{Res}(G) = 1$ then G is residually discrete and hence, its LF-radical is open by Corollary 4·1. Since the LF-radical is characteristic and G is not discrete, we deduce that G is topologically locally finite, hence compact since it is compactly generated.

We assume henceforth that $\mathrm{Res}(G) = G$ and that G is not compact. The above argument shows moreover that G has trivial LF-radical and trivial quasi-centre.

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If G is not totally disconnected, then it is connected. If this is the case, the LF-radical of G is compact (see Lemma 2·2) hence trivial, and we deduce that G is a Lie group. In this case, the standard structure theory of connected Lie groups allows one to show that either $G \cong \mathbf{R}^n$ or G is a direct product of pairwise isomorphic simple Lie groups.

Assume finally that G is totally disconnected. Then Proposition 2·6 guarantees that the set $\mathscr{M}$ of non-trivial minimal closed normal subgroups of G is finite and non-empty. Moreover, since for any proper subset $\mathscr{E} \subset \mathscr{M}$, the subgroup $\overline{\langle M \mid M \in \mathscr{E} \rangle}$ is properly contained in G, it follows that $\mathrm{Aut}(G)$ acts transitively on $\mathscr{M}$.

Now we conclude as in the proof of Theorem E that G is a quasi-product with the elements of $\mathscr{M}$ as topologically simple quasi-factors. □

6. *Composition series with topologically simple subquotients*

We start with an elementary decomposition result on quasi-products.

LEMMA 6·1. *Let G be a locally compact group which is a quasi-product with infinite topologically simple quasi-factors $M_1, \dots, M_n$. Then G admits a sequence of closed normal subgroups*

$$1 = Z_0 < G_1 < Z_1 < G_2 < Z_2 < \cdots < Z_{n-1} < G_n = Z_n = G,$$

where for each $i = 1, \dots, n$, the subgroup G_i is defined as $G_i = \overline{Z_{i-1}M_i}$, the subquotient G_i/Z_{i-1} is topologically simple and $Z_i/G_i = \mathscr{Z}(G/G_i)$.

Proof. The requested properties of the normal series provide in fact a recursive definition for the closed normal subgroups Z_i and G_i. In particular all we need to show is that Z_{i-1} is a maximal proper closed normal subgroup of G_i.

We first claim that $M_j \cap G_{i-1} = 1$ for all $1 \leqslant i < j$, where it is understood that $G_0 = 1$. Since no M_j is Abelian, this amounts to showing that $G_{i-1} \leqslant \mathscr{Z}_G(M_i M_{i+1} \cdots M_n)$. We proceed by induction on i, the base case $i = 1$ being trivial. Now we need to show that M_i and Z_{i-1} are both contained in $\mathscr{Z}_G(M_{i+1} \cdots M_n)$. This is clear for M_i. By induction $M_i \cdots M_n$ maps onto a dense normal subgroup of G/G_{i-1}. Since $Z_{i-1}/G_{i-1} = \mathscr{Z}(G/G_{i-1})$ by definition, we infer that $[Z_{i-1}, M_j] \leqslant G_{i-1}$ for all $j \geqslant i$. Of course we have also $[Z_{i-1}, M_j] \leqslant M_j$ since M_j is normal. The intersection $G_{i-1} \cap M_j$ being trivial by induction, we infer that $[Z_{i-1}, M_j]$ is trivial as well. Thus M_i and Z_{i-1} are indeed both contained in $\mathscr{Z}_G(M_{i+1} \cdots M_n)$, and so is thus G_i. The claim stands proven.

Let now N be a closed normal subgroup G such that $Z_{i-1} \leqslant N < G_i$. By the claim we have $G_i \cap M_j = 1$ for all $j > i$, hence $N \cap M_j = 1$. Now if $N \cap M_i \neq 1$, then $M_i < N$ since M_i is topologically simple. In that case, we deduce that N contains $Z_{i-1}M_i$, which contradicts that N is properly contained in G_i. Thus we have $N \cap M_i = 1$. In particular we deduce that $N \leqslant \mathscr{Z}_G(M_i M_{i+1} \cdots M_n)$. Since $M_i M_{i+1} \ldots M_n$ maps densely into G/G_{i-1}, we deduce that the image of N in G/G_{i-1} is central. By definition, this means that N is contained in Z_{i-1}, thereby proving that Z_{i-1} is indeed maximal normal in G_i. □

Proof of Theorem C. In view of the structure theory of connected locally compact groups (see Lemma 2·2) and of connected Lie groups, the desired result holds in the connected case. Moreover, any homomorphic image of a Noetherian group is itself Noetherian. Therefore, there is no loss of generality in replacing G by the group of components G/G°. Equivalently, we shall assume henceforth that G is totally disconnected.

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We first claim that any closed normal subgroup of G is compactly generated. Indeed, given such a subgroup $N < G$, pick any compact open subgroup U and consider the open subgroup $NU < G$. Since N is a cocompact subgroup of NU, which is compactly generated as G is Noetherian, we infer that N itself is compactly generated, as claimed. Notice that the same property is shared by closed normal subgroups of any open subgroup of G.

Let now $\mathrm{Res}(G)$ denote the discrete residual of G. Thus $G/\mathrm{Res}(G)$ is residually discrete and Noetherian. Corollary 4·1 thus implies that the LF-radical of $G/\mathrm{Res}(G)$ is open, while the above claim guarantees that it is compact. We denote by O the pre-image in G of $\mathrm{Rad}_{\mathscr{LF}}(G/\mathrm{Res}(G))$. Thus O is an open characteristic subgroup of G containing $\mathrm{Res}(G)$ as a cocompact subgroup. In particular, the discrete quotients of O are all finite. Now Theorem F guarantees that $\mathrm{Res}(G) = \mathrm{Res}(O)$ has no non-trivial discrete quotient.

Setting $H = \mathrm{Res}(G)$, we have thus far constructed a series $1 < H < O < G$ of characteristic subgroups with O open and O/H compact. We shall now construct inductively a finite increasing sequence

$$1 = H_0 < H_1 < H_2 < \cdots < H_l = H < O$$

of normal subgroups of O satisfying the following conditions for all $i = 1, \ldots, l$:

(a) If $\mathrm{Rad}_{\mathscr{LF}}(H/H_{i-1})$ is non-trivial, then $\mathrm{Rad}_{\mathscr{LF}}(G/H_i) = 1$;

(b) H_i/H_{i-1} is either compact, or isomorphic to $\mathbf{Z}^n$ for some n, or to a quasi-product with topologically simple pairwise O-conjugate quasi-factors.

Let $j > 0$ and assume that the first $j-1$ terms $H_0, \ldots, H_{j-1}$ of the desired series have already been constructed, in such a way that properties (a) and (b) hold with $i < j$. We proceed to define H_j as follows.

If $\mathrm{Rad}_{\mathscr{LF}}(H/H_{j-1})$ is non-trivial, then we let H_j be the pre-image in H of $\mathrm{Rad}_{\mathscr{LF}}(H/H_{j-1})$. Properties (a) and (b) clearly hold for $j = i$ in this case.

Assume now that $\mathrm{Rad}_{\mathscr{LF}}(H/H_{j-1}) = 1$ and that $\mathscr{QZ}(H/H_{j-1})$ is non-trivial. Let M denote the closure of $\mathscr{QZ}(H/H_{j-1})$ in H/H_{j-1}. Thus M is characteristic and quasi-discrete. Furthermore, the fact that $\mathrm{Rad}_{\mathscr{LF}}(H/H_{j-1}) = 1$ implies that M has no non-trivial compact normal subgroup. Therefore Corollary 4·4 ensures that $M = \mathscr{QZ}(H/H_{j-1})$ and that the identity component M° is open and isomorphic to $\mathbf{R}^n$ for some n.

Since H is totally disconnected, it follows that M° is trivial. Thus M is totally disconnected as well, hence compact-by-discrete in view of Proposition 4·3. But M has no non-trivial compact normal subgroup since $\mathrm{Rad}_{\mathscr{LF}}(H/H_{j-1})$ is trivial and it follows that M is discrete. We claim that M is Abelian. Indeed, since M is discrete and finitely generated, its centraliser in H/H_{j-1} is open. By assumption H has no non-trivial discrete quotient, and this property is inherited by the quotient H/H_{j-1}. We deduce that $\mathscr{Z}_{H/H_{j-1}}(M) = H/H_{j-1}$; in other words M is central in H/H_{j-1} hence Abelian, as claimed. Let M_0 denote the unique maximal free Abelian subgroup of M. Then M_0 is non-trivial since M is not compact. We define H_j to be the pre-image of M_0 in H. Then H_j is characteristic and again, properties (a) and (b) are both satisfied with $i = j$ in this case.

It remains to define H_j in the case where the LF-radical $\mathrm{Rad}_{\mathscr{LF}}(H/H_{j-1})$ and the quasi-centre $\mathscr{QZ}(H/H_{j-1})$ are both trivial. In that case, Proposition 2·6 guarantees that H/H_{j-1} contains some non-trivial minimal closed normal subgroups of O/H_{j-1}, say M, provided H/H_{j-1} is non-trivial. Clearly M is characteristically simple, so that, by Corollary D, it is a quasi-product with finitely many topologically simple quasi-factors. Now O acts transitively by conjugation on these quasi-factors; otherwise M would contain a proper closed normal

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subgroup (see Proposition 2·6), contradicting minimality. It remains to define H_j as the pre-image of M in O.

Hence (a) and (b) hold with $i = j$ in all cases.

We have thus constructed an ascending chain of subgroups $1 = H_0 < H_1 < H_2 < \cdots < H < O$ which are all normal in O and we proceed to show that $H_k = H$ for some large enough index k. Suppose for a contradiction that this is not the case and set $H_\infty = \overline{\bigcup_{i=1}^{\infty} H_i}$. Since H_∞ is normal in O, it is compactly generated (see the second paragraph of the present proof above). Let $V < H_\infty$ be a compact open subgroup. Then the ascending chain $V \cdot H_1 < V \cdot H_2 < \cdots$ yields a covering of H_∞ by open subgroups. The compact generation of H_∞ thus implies that $V \cdot H_k = H_\infty$ for k large enough. In particular H_k is cocompact in H_∞. Therefore $\mathrm{Rad}_{\mathscr{LF}}(H/H_k)$ is non-trivial. By property (a), this implies that $\mathrm{Rad}_{\mathscr{LF}}(H/H_{k+1})$ is trivial and hence $H_\infty \subseteq H_{k+1}$. This contradiction establishes the claim.

It only remains to show that the series of characteristic subgroups $1 = H_0 < H_1 < H_2 < \cdots < H_l = H$ that we have constructed can be refined into a subnormal series satisfying the desired conditions on the subquotients. By construction, it suffices to refine the non-compact non-Abelian subquotients H_i/H_{i-1}. Since these are quasi-products with finitely many topologically simple pairwise O-conjugate quasi-factors, we may replace O by an appropriate closed normal subgroup of finite index, say O', in such a way that for all i, each topologically simple quasi-factor of H_i/H_{i-1} is normal in O'/H_{i-1}. Consider now the decomposition of H_i/H_{i-1} provided by Lemma 6·1. Each term of this decomposition is normal in O'/H_{i-1}, and must therefore be compactly generated. Therefore, the corresponding subquotients are compactly generated. In particular, the Abelian subquotients are compact-by-$\mathbf{Z}^k$. Introducing these intermediate terms in the series $1 = H_0 < H_1 < H_2 < \cdots < H_l = H < O' < O < G$, we obtain a refinement which has all the desired properties. □

Appendix I. The adjoint closure and asymptotically central sequences

On the Braconnier topology. Let G be a locally compact group and $\mathrm{Aut}(G)$ denote the group of all homeomorphic automorphisms of G. There is a natural topology, sometimes called the **Braconnier topology**, turning $\mathrm{Aut}(G)$ into a Hausdorff topological group; it is defined by the sub-base of identity neighbourhoods

$$\mathfrak{A}(K, U) := \left\{\alpha \in \mathrm{Aut}(G) \mid \forall x \in K,\ \alpha(x)x^{-1} \in U \text{ and } \alpha^{-1}(x)x^{-1} \in U\right\},$$

where $K \subseteq G$ is compact and $U \subseteq G$ is an identity neighbourhood (see [**Bra48**, Chapter IV, Section 1] or [**HR79**, Theorem 26·5]).

In other words, this topology is the common refinement of the compact-open topology for automorphisms and their inverses; recall in addition that a topological group has canonical uniform structures so that the compact-open topology coincides with the topology of uniform convergence on compact sets ([**Bra48**, p. 59] or [**Kel75**, Section 7·11]).

In fact, the Braconnier topology coincides with the restriction of the g-topology on the group of all homeomorphisms of G introduced by Arens [**Are46**], itself hailing from Birkhoff's C-convergence [**Bir34**, Section 11]. It can alternatively be defined by restricting the compact-open topology for the Alexandroff compactification of G, an idea originating with van Dantzig and van der Waerden [**vDvdW28**, Section 6].

Braconnier shows by an example that the compact-open topology itself is in general too coarse to turn $\mathrm{Aut}(G)$ into a topological group [**Bra48**, pp. 57–58]. We shall establish below a basic dispensation from this fact for the adjoint representation (Proposition I·1). Meanwhile, we recall that the Braconnier topology coincides with the compact-open topology when G is compact ([**Are46**, Lemma 1]) and when G is locally connected ([**Are46**,

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Theorem 4]). There are of course non-locally-connected connected groups: the solenoids of Vietoris [**Vie27**, II] and van Dantzig [**vD30**, Section 2, Satz 1]. Nevertheless, using notably the solution to Hilbert's fifth problem, S.P. Wang showed that the two topologies still coincide for all connected and indeed almost connected locally compact groups [**Wan69**, Corollary 4·2]. Finally, the topologies coincide for G discrete and $G = \mathbf{Q}_p^n$, see [**Bra48**, p. 58].

We emphasise that the Braconnier topology on $\mathrm{Aut}(G)$ need not be locally compact, see [**HR79**, Section 26·18·k]. A criterion ensuring that $\mathrm{Aut}(G)$ is locally compact will be presented in Theorem I·6 below in the case of totally disconnected groups.

Nevertheless, $\mathrm{Aut}(G)$ is a Polish (hence Baire) group when G is second countable. Indeed, it is by definition closed (even for the weaker pointwise topology) in the group of homeomorphisms of G endowed with Arens' g-topology; the latter is second countable (see e.g. [**GP57**, 5·4]) and complete for the *bilateral* uniform structure [**Are46**, Theorem 6]. Notice that this complete uniformisation is not the usual left or right uniform structure, which is known to be sometimes incomplete at least for the group of homeomorphisms (Arens, *loc. cit.*).

The Baire property implies for instance that $\mathrm{Aut}(G)$ is discrete when countable, which was observed in [**Pla77**, Satz 2] for G itself discrete.

Adjoint representation. Given a closed normal subgroup $N < G$, the conjugation action of G on N yields a map $G \to \mathrm{Aut}(N)$ which is continuous (see [**HR79**, Theorem 26·7]). In particular, the natural map $\mathrm{Ad}\colon G \to \mathrm{Aut}(G)$ induced by the conjugation action is a continuous homomorphism. We endow the group $\mathrm{Ad}(G) < \mathrm{Aut}(G)$ with the Braconnier topology. Thus, a sub-base of identity neighbourhoods is given by the image in $\mathrm{Ad}(G)$ of all subsets of G of the form

$$\mathfrak{B}(K, U) := \big\{ g \in G \mid [g, K] \subseteq U \text{ and } [g^{-1}, K] \subseteq U \big\},$$

where (K, U) runs over all pairs of compact subsets and identity neighbourhoods of G.

As an abstract group, $\mathrm{Ad}(G)$ is isomorphic to $G/\mathscr{Z}(G)$; we emphasise however that the latter is endowed with the generally finer quotient topology.

PROPOSITION I·1. *Let G be a locally compact group such that the group of components G/G° is unimodular.*

Then the Braconnier topology on $\mathrm{Ad}(G)$ *coincides with the compact-open topology.*

Proof. Let $\{g_\alpha\}_\alpha$ be a net in G such that $\mathrm{Ad}(g_\alpha)$ converges to the identity in the compact-open topology. According to the result of S.P. Wang quoted earlier in this section, the automorphisms $\mathrm{Ad}(g_\alpha)|_{G^\circ}$ of the identity component G° converge to the identity for the Braconnier topology on $\mathrm{Aut}(G^\circ)$. According to [**Wan69**, Proposition 2·3], it now suffices to prove that the induced automorphisms on G/G° also converge to the identity for the Braconnier topology on $\mathrm{Aut}(G/G^\circ)$. Therefore, we can suppose henceforth that G is totally disconnected.

By assumption, $\{g_\alpha\}$ eventually penetrates every set of the form

$$\mathfrak{B}'(K', U') := \{ g \in G \mid [g, K'] \subseteq U' \},$$

where $K' \subseteq G$ is compact and $U' \subseteq G$ is a neighbourhood of $e \in G$. Thus it suffices to show that for all $K \subseteq G$ compact and $U \subseteq G$ identity neighbourhood, there is K' and U' with

$$\mathfrak{B}'(K', U') \subseteq \mathfrak{B}(K, U).$$

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Since G is totally disconnected, there is a compact open subgroup $U' < G$ contained in U. Set $K' = K \cup U'$ and fix any $g \in \mathfrak{B}'(K', U')$. We need to show that $[g^{-1}, K] \subseteq U$.

First, notice that $[g^{-1}, K] = g^{-1}[g, K]^{-1}g$. Next, $[g, K]$ and hence also $[g, K]^{-1}$ is in U'. Finally, $[g, U'] \subseteq U'$ means that $gU'g^{-1} \subseteq U'$; by unimodularity, it follows that g normalises U'. We conclude that $[g^{-1}, K] \subseteq U' \subseteq U$, as was to be shown. $\square$

A locally compact group G for which the map Ad: $G \to \mathrm{Ad}(G)$ is closed will be called **Ad-closed**. In that case, $\mathrm{Ad}(G)$ is isomorphic to $G/\mathscr{Z}(G)$ as a topological group and thus in particular it is locally compact.

The group G can fail to be Ad-closed even when it is a connected Lie group (Example I·3 below; see also e.g. [**LW70**, **Zer76**]). Perhaps more strikingly, G can fail to be Ad-closed even when countable, discrete and $\mathbf{Z}/2\mathbf{Z}$-by-abelian [**Wu71**, 4·5].

Asymptotically central sequences. Let G be a locally compact group. A sequence $\{g_n\}$ of elements of G is called **asymptotically central** if $\mathrm{Ad}(g_n)$ converges to the identity in $\mathrm{Ad}(G)$. Obvious examples are central sequences or sequences converging to e; we shall investigate the existence of non-obvious ones (for an admittedly limited analogy, compare the *property* Γ introduced for II_1-factors by Murray and von Neumann, [**MvN43**, Definition 6·1·1]).

The existence of suitably non-trivial asymptotically central sequences is related to the question whether the Braconnier topology on G (strictly speaking, on $G/\mathscr{Z}(G)$) coincides with the initial topology, as follows.

PROPOSITION I·2. *Let G be a second countable locally compact group. The following conditions are equivalent*:

(i) $\mathrm{Ad}(G)$ *is locally compact*;

(ii) *The continuous homomorphism* $\mathrm{Ad} : G \to \mathrm{Ad}(G)$ *is closed*;

(iii) *The map* $G/\mathscr{Z}(G) \to \mathrm{Ad}(G)$ *is a topological group isomorphism*;

(iv) *The image in $G/\mathscr{Z}(G)$ of every asymptotically central sequence is relatively compact.*

A sufficient condition for this is that G admits some compact open subgroup U such that $\mathscr{N}_G(U)$ is compact.

Proof. (i) $\Rightarrow$ (ii) This is a well-known application of the Baire category principle, going back at least to [**Pon39**, Theorem XIII].

(ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) follow from the definitions.

(iv) $\Rightarrow$ (i) Let $\{K_n\}$ be an increasing sequence of compact subsets of G whose union covers G and let $\{U_n\}$ be a decreasing family of sets providing a basis of neighbourhoods of $e \in G$. Assuming for a contradiction that $\mathrm{Ad}(G)$ is not locally compact, none of the sets $\mathfrak{B}(K_n, U_n)$ can have a relatively compact image in $\mathrm{Ad}(G)$. Therefore, we can choose for each n an element g_n in $\mathfrak{B}(K_n, U_n)$ but not in $K_n.\mathscr{Z}(G)$. By construction, the sequence $\{g_n\}$ is unbounded in $G/\mathscr{Z}(G)$ but $\mathrm{Ad}(g_n)$ converges to the identity, a contradiction.

Finally, notice that if U is a compact open subgroup of G, then $\mathscr{N}_G(U) = \mathfrak{B}(U, U)$. This shows that if $\mathscr{N}_G(U)$ is compact, then $\mathrm{Ad}(G)$ admits $\mathrm{Ad}(\mathfrak{B}(U, U))$ as a compact identity neighbourhood. $\square$

We recall from [**KK44**] that a σ-compact locally compact group G always possesses a compact normal subgroup Q such that the quotient G/Q is metrisable. In particular, any compactly generated locally compact group without non-trivial compact normal subgroup satisfies the hypotheses of Proposition I·2.

The following construction provides examples of Lie groups which are not Ad-closed.

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Example I·3. Let $L \cong \mathbf{R} < \mathbf{R}^2$ be a one-parameter subgroup with irrational slope and denote by Z the image of L in the torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$. Thus Z is a connected dense subgroup of $\mathbf{T}^2$. Let us now choose a continuous faithful representation of $\mathbf{T}^2$ in $O(4)$ and consider the corresponding semi-direct product $H = \mathbf{T}^2 \ltimes \mathbf{R}^4$. We define $G = Z \ltimes \mathbf{R}^4$. Thus G is a connected subgroup of the Lie group $O(4) \ltimes \mathbf{R}^4$.

We claim that G is not Ad-closed. Indeed, let (z_n) be an unbounded sequence of elements of Z which converge to 1 in the torus $\mathbf{T}^2$. One verifies easily that G is centrefree and that the above sequence is asymptotically central in G. This yields the desired claim in view of Proposition I·2.

An illustration of the relevance of the notion of Ad-closed groups is provided by the following.

LEMMA I·4. *Let G be a locally compact group and $H < G$ be a closed subgroup. If H is* Ad*-closed, then $H.\mathscr{Z}_G(H)$ is closed in G.*

Proof. Without loss of generality, we may assume that $H.\mathscr{Z}_G(H)$ is dense in G. Then H is normal and there is a continuous conjugation action $\alpha : G \to \mathrm{Aut}(H)$. Since $H.\mathscr{Z}_G(H)$ is dense, it follows that $\mathrm{Ad}(H)$ is dense in $\alpha(G)$. Now $\mathrm{Ad}(H)$ being closed in $\mathrm{Aut}(H)$ by hypothesis, we infer that $\alpha(G) = \mathrm{Ad}(H)$. The result follows, since the pre-image of $\mathrm{Ad}(H)$ in G is nothing but $H.\mathscr{Z}_G(H)$. □

The adjoint closure. The closure of $\mathrm{Ad}(G)$ in $\mathrm{Aut}(G)$ will be called the **adjoint closure** of G and will be denoted by $\overline{\mathrm{Ad}(G)}$. We think of an automorphism in $\overline{\mathrm{Ad}(G)}$ as "approximately inner". We point out that $\mathrm{Ad}(G)$ is normal in $\mathrm{Aut}(G)$ and hence in particular in $\overline{\mathrm{Ad}(G)}$.

Basic properties of the adjoint closure are summarised in the following. Notice that $\overline{\mathrm{Ad}(G)}$ is not assumed locally compact except in the last item.

LEMMA I·5. *Let G be a locally compact group.*

(i) *If G is centrefree, then so is $\overline{\mathrm{Ad}(G)}$.*
(ii) *If G is topologically simple, then so is $\overline{\mathrm{Ad}(G)}$.*
(iii) *If G is totally disconnected, then so is $\overline{\mathrm{Ad}(G)}$.*
(iv) *Suppose that $\overline{\mathrm{Ad}(G)}$ is locally compact. If G is compactly generated, then so is $\overline{\mathrm{Ad}(G)}$.*

Proof. (i) Given $\alpha \in \mathscr{Z}_{\mathrm{Aut}(G)}(\mathrm{Ad}(G))$, we have $\alpha(g)x\alpha(g)^{-1} = gxg^{-1}$ for all $g, x \in G$. Thus $\alpha(g)^{-1}g$ belongs to $\mathscr{Z}(G)$ and the result follows.
(ii) Let $H < \overline{\mathrm{Ad}(G)}$ be a closed normal subgroup and let $H_0 = \mathrm{Ad}^{-1}(H)$ be the pre-image of H in G. Then H_0 is a closed normal subgroup of G and is thus trivial of the whole group. If $H_0 = G$, then H contains $\mathrm{Ad}(G)$ which is dense, thus $H = G$ as well. If $H_0 = 1$, then $H \cap \mathrm{Ad}(G) = 1$. This implies that $[H, \mathrm{Ad}(G)] \subseteq H \cap \mathrm{Ad}(G) = 1$. Thus H commutes with the dense subgroup $\mathrm{Ad}(G)$ and is thus contained in the centre of $\overline{\mathrm{Ad}(G)}$, which is trivial by the assertion (i).
(iii) See [**Bra48**, IV Section 2] or [**HR79**, Theorem 26·8].
(iv) Let U be a compact neighbourhood of the identity in $\overline{\mathrm{Ad}(G)}$ and $C \subseteq G$ a compact generating set. Then $U.\,\mathrm{Ad}(C)$ generates $\overline{\mathrm{Ad}(G)}$. □

Locally finitely generated groups. We shall say that a totally disconnected locally compact group G is **locally finitely generated** if G admits some compact open subgroup that is

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topologically finitely generated, i.e. possesses a finitely generated dense subgroup. Since any two compact open subgroups of G are commensurable, it follows that G is locally finitely generated if and only if *any* compact open subgroup is topologically finitely generated. Examples of such include p-adic analytic groups (see e.g. [**DdSMS99**, Theorem 8·36]), many complete Kac–Moody groups over finite fields [**CER08**, Theorem 6·4] as well as several (but not all) locally compact groups acting properly on locally finite trees [**Moz98**].

An important property of finitely generated profinite groups is that they admit a (countable) basis of identity neighbourhoods consisting of characteristic subgroups, because they have only finitely many closed subgroups of any given index. Locally finitely generated groups are thus covered by the following result.

THEOREM I·6. *Let G be a totally disconnected compactly generated locally compact group. Suppose that G admits an open subgroup U that has a basis of identity neighbourhoods consisting of characteristic subgroups of U (e.g. G is locally finitely generated).*

Then $\mathrm{Aut}(G)$ *is locally compact.*

The proof will use the following version of the Arzelà–Ascoli Theorem.

PROPOSITION I·7. *Let G be a locally compact group and $V \subseteq \mathrm{Aut}(G)$ a subset such that*:

(i) $V = V^{-1}$;

(ii) *G has arbitrarily small V-invariant identity neighbourhoods*;

(iii) *$V(x)$ is relatively compact in G for each $x \in G$.*

Then V is relatively compact in $\mathrm{Aut}(G)$.

In the case where V is a compact subgroup of $\mathrm{Aut}(G)$, this is [**GM67**, Theorem 4·1].

Proof of Proposition I·7. Point (ii) implies that V is equicontinuous (in fact, uniformly equicontinuous). Therefore, we can apply Arzelà–Ascoli (in the generality of [**Bou74**, X Section 2, no. 5]) and deduce that V has compact closure in the space of continuous maps $G \to G$ endowed with the compact-open topology (which, as mentioned, coincides with the topology of compact convergence). The closure of V remains in the space of continuous endomorphisms since the latter is closed even pointwise. In view of the symmetry of the assumptions and of the fact that composition is continuous in the compact-open topology [**Dug66**, XII·2·2], the closure of V remains in $\mathrm{Aut}(G)$ and is compact for the Braconnier topology. □

Proof of Theorem I·6. Let $U < G$ be an open subgroup admitting a basis of identity neighbourhoods $\{U_\alpha\}_\alpha$ consisting of characteristic subgroups of U. We can assume U compact upon intersecting with a compact open subgroup. Let $C \subseteq G$ be a symmetric compact set generating G and containing U. We shall prove that $V := \mathfrak{A}(C, U) \subseteq \mathrm{Aut}(G)$ satisfies the assumptions of Proposition I·7; this then establishes the theorem.

The first assumption holds by definition. For the second, notice first that V normalises U since $U \subseteq C$ implies

$$V \subseteq \mathfrak{A}(U, U) = \mathscr{N}_{\mathrm{Aut}(G)}(U).$$

Assumption (ii) holds since the identity neighbourhoods U_α are characteristic, hence normalised by $\mathscr{N}_{\mathrm{Aut}(G)}(U)$.

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In order to establish the last assumption, choose $x \in G$. Since C is generating and symmetric, there is an integer d such that $x \in C^d$. The definition of V shows that for any automorphism $n \in V$, we have $n(x) \in (U.C)^d$; this implies (iii). □

The adjoint closure of discrete groups. A particularly simple illustration of the concepts introduced above is provided by discrete groups. The Braconnier topology on $\mathrm{Aut}(G)$ is then the topology of pointwise convergence and coincides with pointwise convergence of the inverse. The adjoint closure $\overline{\mathrm{Ad}(G)}$ coincides therefore with the group $\mathrm{Linn}(G)$ of **locally inner** automorphisms, i.e. automorphisms that coincide on every finite set with some inner automorphism. This concept was apparently first introduced (локально внутренним) by Gol'berg [Gb46, § 3 Определение 5]

Here are a few elementary properties of the resulting correspondance $G \mapsto \overline{\mathrm{Ad}(G)}$ from abstract (resp. countable) groups to topological (resp. Polish) groups.

PROPOSITION I·8. *Let G be a discrete group and $A = \overline{\mathrm{Ad}(G)} = \mathrm{Linn}(G)$ its adjoint closure.*

(i) $\mathrm{Ad}(G) \leqslant \mathscr{QZ}(A)$. *In particular A is quasi-discrete; it is discrete if and only if there is a finite set $F \subseteq G$ with $\mathscr{Z}_G(F) = \mathscr{Z}(G)$.*

(ii) *A is compact if and only if G is FC.*[1]

(iii) *A is locally compact if and only if there is a finite set $F \subseteq G$ such that the index $[\mathscr{Z}_G(F) \colon \mathscr{Z}_G(F')]$ is finite for every finite $F' \supseteq F$ in G.*

(iv) *A is locally compact and compactly generated if and only if there is F as in* (iii) *and $F_0 \subseteq G$ finite such that $F_0 \cup \mathscr{Z}_G(F)$ generates G.*

Proof. All verifications are straightforward. One uses notably an elementary version of Proposition I·7 stating that, for G discrete, a subset $V \subseteq \mathrm{Aut}\, G$ has compact closure if and only if $V(x)$ is finite for all $x \in G$. □

Discrete groups are a safe playground to experiment with intermediate topologies inbetween the original topology and the Braconnier topology induced *via* the adjoint representation. The following construction will lead to interesting examples, see Appendix II and especially Example II·7.

Let N be a discrete group and $U < N$ a subgroup such that:

(i) $\mathscr{Z}_U(g)$ has finite index in U for all $g \in N$;

(ii) the intersection of all N-conjugates of U is trivial.

In particular, the N-conjugates of U generate a completable group topology [**Bou60**, TG III, Section 3, no 4] and N injects into the resulting complete totally disconnected topological group M.

PROPOSITION I·9.

(i) *The group M is locally compact; in fact U has compact-open closure in M.*

(ii) *There is a (necessarily unique) continuous injective homomorphism $M/\mathscr{Z}(M) \to \overline{\mathrm{Ad}(N)}$ compatible with the maps $N \to M$ and $N \to \overline{\mathrm{Ad}(N)}$. In particular, the dense image of N in M is normal and quasi-central (thus M is quasi-discrete).*

(iii) *If N is centrefree (resp. simple), then $M/\mathscr{Z}(M)$ is centrefree (resp. topologically simple).*

(iv) $M/\mathscr{Z}(M) = \overline{\mathrm{Ad}(N)}$ *if and only if $\mathscr{Z}_N(F) \subseteq U$ for some finite $F \subseteq N$.*

[1] Recall that G is an **FC-group** if all its conjugacy classes are finite.

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Proof. The first assertion is due to the fact that the closure of U in M is a quotient of the profinite completion of U. The second assertion follows from the fact that a system of neighbourhoods of the identity for the M-topology on N is given by $U \cap V$, where V ranges over the $\overline{\mathrm{Ad}(N)}$-neighbourhoods of the identity. The third assertion follows by the same argument as in the proof of Lemma I·5. For the last assertion, observe that $M = \overline{\mathrm{Ad}(N)}$ if and only if U is open in the Braconnier topology on N. □

Example I·10. Let Ω be a countably infinite set, let $N < \mathrm{Sym}(\Omega)$ be an infinite (almost) simple group of alternating **finitary permutations**, i.e. permutations with finite support. Choose also an equivalence relation $\sim$ on Ω all of whose equivalence classes have finite cardinality, and let $U < N$ be an infinite subgroup preserving each equivalence class. For each $g \in N$, there is a finite index subgroup $U' < U$ such that g and U' have disjoint support. Therefore $\mathscr{Z}_U(g)$ has finite index in U for all $g \in N$. Moreover the intersection of all N-conjugates of U is trivial since N is almost simple.

Concretely, one could define N as the group of all alternating finitary permutation, which is indeed simple, and define U as the subgroup preserving all equivalence classes of a relation $\sim$ which is a partition into subsets of fixed size $k > 1$. The group U is then isomorphic to a restricted direct sum of finite alternating groups of degree k and M is a totally disconnected locally compact group which is topologically simple, topologically locally finite and quasi-discrete. (Here M is centrefree because every asymptotically central sequence of N converges pointwise to the identity.)

We remark that the examples of topologically simple locally compact groups admitting a dense normal subgroup which were constructed Willis in [**Wil07**, Section 3] all fit in this set-up, and can all be obtained by taking various specializations of the groups $N < \mathrm{Sym}(\Omega)$ and $U < N$.

Remark I·11. The previous example takes advantage of the fact that the group of all finitary permuations of a countably infinite set Ω is not Ad-closed. Notice however that its adjoint closure, which incidentally coincides with the group $\mathrm{Sym}(\Omega)$ of *all* permutations of Ω, is however not locally compact. Proposition I·9 and Example I·10 thus correspond to completions which are genuinely intermediate between $\mathrm{Ad}(N)$ and $\overline{\mathrm{Ad}(N)}$. This is an instance of a general scheme that we shall present below, see Proposition II·5.

Appendix II. Quasi-products and dense normal subgroups

For general locally compact groups, there is a naturally occurring structure that is weaker than direct products. We establish its basic properties and give some examples. In order to avoid some obvious degeneracies, it is good to have in mind the centrefree case.

Definitions and the Galois connection. Let G be a topological group. We call a closed normal subgroup $N \lhd G$ a **quasi-factor** of G if $N.\mathscr{Z}_G(N)$ is dense in G. In other words, this means that the G-action on N is “approximately inner” in the sense that the image of G in $\mathrm{Aut}(N)$ is contained in the adjoint closure $\overline{\mathrm{Ad}(N)}$.

If N is a quasi-factor, then $N \cap \mathscr{Z}_G(N)$ is contained in the centre of G. Thus, in the centrefree case, quasi-factors provide an example of the following concept with $p = 2$:

We say that G is the **quasi-product** of the closed normal subgroups $N_1, \ldots, N_p$ if the multiplication map

$$N_1 \times \cdots \times N_p \longrightarrow G$$

is injective with dense image. We call the groups N_i the **quasi-factors** of this quasi-product;

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notice that N_i and N_j commute for all $i \neq j$ and therefore each N_i is indeed a quasi-factor in the earlier sense.

Given a quasi-product, one has a family of quotients $G \twoheadrightarrow S_i$ defined by $S_i = G/\mathscr{Z}_G(N_i)$. Notice that the image of N_i in S_i is a dense normal subgroup; moreover, when G is centrefree, N_i injects into S_i. Therefore, we obtain an injection with dense image:

$$G/\mathscr{Z}(G) \longrightarrow S_1 \times \cdots \times S_p.$$

(The relation between quasi-products and dense normal subgroups will be further investigated below; see Example II·7.)

The map $N \mapsto \mathscr{Z}_G(N)$ is an antitone Galois connection on the set of closed normal subgroups of G and in particular also on the collection of quasi-factors. It turns out that this correspondence behaves particularly well for certain groups appearing in the main results of this article, as follows. Denote by **Max** (resp. **Min**) the set of all maximal (resp. minimal) closed normal subgroups which are non-trivial.

PROPOSITION II·1. *Let G be a non-trivial compactly generated totally disconnected locally compact group. Assume that G is centrefree and without non-trivial discrete quotient. If $\bigcap$* **Max** *is trivial, then the following hold*:

(i) **Min** *and* **Max** *are finite and non-empty*;
(ii) *the assignment $N \mapsto \mathscr{Z}_G(N)$ defines a bijective correspondence from* **Min** *to* **Max**;
(iii) *every element of* **Min** $\cup$ **Max** *is a quasi-factor*;
(iv) *G is the quasi-product of its minimal normal subgroups.*

This result provides in particular additional information on characteristically simple groups, which supplements Corollary D. Indeed, the hypotheses of the proposition are in particular fulfilled by characteristically simple groups falling in case (iv) of Corollary D (see also Proposition 5·4).

Proof of Proposition II·1 Notice that G has no non-trivial compact quotient since every discrete quotient is trivial. The set **Max** is non-empty since it has trivial intersection; its finiteness follows from Theorem A. Moreover, G embeds in the product of simple groups $\prod_{K \in \mathbf{Max}} G/K$, which implies that G has trivial quasi-centre and no non-trivial compact normal subgroup. Indeed, otherwise some G/K would be compact or quasi-discrete (as in the proof of Corollary 5·1). In particular, Theorem B implies that **Min** is finite and non-empty; assertion (i) is established. Actually, we shall use below not only that **Min** is non-empty, but that every non-trivial closed normal subgroup of G contains a minimal one, see Proposition 2·6.

For the duration of the proof, denote by $\mathbf{Max}_{\mathrm{QF}}$ the subset of **Max** consisting of those elements which are quasi-factors of G.

We claim that the map $N \mapsto \mathscr{Z}_G(N)$ defines a one-to-one correspondence of **Min** *onto* $\mathbf{Max}_{\mathrm{QF}}$. *Moreover, every element of* **Min** $\cup$ $\mathbf{Max}_{\mathrm{QF}}$ *is a quasi-factor.*

Let $N \in$ **Min**. By hypothesis there is some $K \in$ **Max** which does not contain N. By minimality of N we deduce that $N \cap K$ is trivial and hence $[N, K] = 1$. Therefore $N.K$ is dense in G by maximality of K. In particular, N and K are both quasi-factors. Moreover, since $\mathscr{Z}_G(N)$ contains K, maximality implies $K = \mathscr{Z}_G(N)$ because G is centrefree. In other words, $N \mapsto \mathscr{Z}_G(N)$ defines a map **Min** $\to$ $\mathbf{Max}_{\mathrm{QF}}$. Since any minimal closed normal subgroup of G different from N commutes with N, it is contained in K.

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Therefore, the above map is an injection of **Min** into $\mathbf{Max}_{\mathrm{QF}}$. It remains to show that it is surjective.

To this end, pick $K \in \mathbf{Max}_{\mathrm{QF}}$. Then $\mathscr{Z}_G(K)$ is a non-trivial closed normal subgroup of G. It therefore contains an element of **Min**, say N. By definition K is contained in $\mathscr{Z}_G(N)$, whence $K = \mathscr{Z}_G(N)$ by maximality. The claim stands proven.

We claim that every element of $\mathbf{Max} \setminus \mathbf{Max}_{\mathrm{QF}}$ *contains every element of* **Min**.

If $K \in \mathbf{Max}$ and $N \in \mathbf{Min}$ are such that $N \not\leqslant K$, then N and K commute and, hence, $K = \mathscr{Z}_G(N)$. Thus $K \in \mathbf{Max}_{\mathrm{QF}}$ by the previous claim.

We claim that $\bigcap_{K \in \mathbf{Max}_{\mathrm{QF}}} K = 1$.

Otherwise $\bigcap_{K \in \mathbf{Max}_{\mathrm{QF}}} K$ would contain some $N \in \mathbf{Min}$, which is also contained in every $L \in \mathbf{Max} \setminus \mathbf{Max}_{\mathrm{QF}}$ by the previous claim. This contradicts the hypothesis that $\bigcap \mathbf{Max}$ is trivial.

We claim that $G = \overline{[G, G]}$.

By hypothesis G has no non-trivial discrete quotient. This property is clearly inherited by any quotient of G. Therefore, the claim follows from the fact that the only totally disconnected locally compact Abelian group with that property is the trivial group.

We claim that $G = \overline{\langle N \mid N \in \mathbf{Min}\rangle}$.

Set $H = \overline{\langle N \mid N \in \mathbf{Min}\rangle}$ and $A = G/H$. It follows from the first claim above that every element $K \in \mathbf{Max}_{\mathrm{QF}}$ has dense image in A. In view of the third claim above, we infer that A admits dense normal subgroups $L_1, \dots, L_p$ with trivial intersection. In view of Lemma II·2 below, it follows that A is nilpotent, hence trivial by the previous claim.

We claim that $\mathbf{Max} = \mathbf{Max}_{\mathrm{QF}}$.

Indeed, every element of $\mathbf{Max} \setminus \mathbf{Max}_{\mathrm{QF}}$ contains every element of **Min**. By the previous claim, this implies that every element of $\mathbf{Max} \setminus \mathbf{Max}_{\mathrm{QF}}$ coincides with G and thus fails to be a non-trivial subgroup.

Now assertions (ii), (iii) and (iv) follow at once. □

LEMMA II·2. *Let A be a Hausdorff topological group containing p dense normal subgroups $L_1, \dots, L_p$ such that $\bigcap_{i=1}^{p} L_i = 1$. Then A is nilpotent of degree $\leqslant p - 1$*

Proof. For each $j = 1, \dots, p$, we set $M_j = \bigcap_{i=j}^{p} L_i$. In particular M_1 is trivial and $M_p = L_p$.

Set $A_i = A/\overline{M_i}$ for all $i = 1, \dots, p$. We have a chain of continuous surjective maps

$$A \cong A_1 \to A_2 \to \cdots \to A_{p-1} \to A_p \cong 1.$$

Let $i < p$. Since $M_i = L_i \cap M_{i+1}$, it follows that the respective images of L_i and M_{i+1} in A_i commute. Since moreover L_i is dense in A, it maps densely in A_i and we deduce that the image of M_{i+1} in A_i is central. In particular A_i is a central extension of A_{i+1}. It readily follows that the upper central series of A terminates after at most $p - 1$ steps. □

On the non-Hausdorff quotients of a quasi-product. The following result describes the *algebraic* structure of the generally non-Hausdorff quotient $G/N_1 \cdots N_p$ (its topological structure being trivial). It applies in particular to the case of totally disconnected groups that are Noetherian.

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PROPOSITION II·3. *Let G be a totally disconnected locally compact group that is a quasi-product with quasi-factors $N_1, \dots, N_p$.*

(i) *If N_i possesses a maximal compact subgroup U_i for some $i \in \{1, \dots, p\}$, then for each compact subgroup $U < G$ containing U_i, the quotient $U/U_i.(U \cap \mathscr{Z}_G(N_i))$ is Abelian.*

(ii) *If N_i possesses a maximal compact subgroup U_i for each $i \in \{1, \dots, p\}$, then the quotient $G/\mathscr{Z}(G).N_1 \cdots N_p$ is Abelian.*

Proof. Let $U_i < N_i$ be a maximal compact subgroup and let $U < G$ be a compact open subgroup containing U_i. Since $U \cap N_i$ is a compact subgroup of N_i containing U_i, we have $U \cap N_i = U_i$ by maximality. Let also $Z_i = \mathscr{Z}_G(N_i)$. We shall first show that $U/U_i.(U \cap Z_i)$ is Abelian, which is the assertion (i).

In order to establish this, consider the open subgroup $H_i := U.N_i$. Since U_i is a maximal compact subgroup of N_i, it follows that U is a maximal compact subgroup of H_i.

Notice that $H_i \cap Z_i$ is centralised by a cocompact subgroup of H_i, namely N_i. Therefore $H_i \cap Z_i$ is a topologically $\overline{\mathrm{FC}}$-group, indeed the H_i-conjugacy class of every element is relatively compact. By Ušakov's result [**Uša63**] (see Theorem 2·3 above), the set Q of all its periodic elements coincides with the LF-radical and the quotient $(H_i \cap Z_i)/Q$ is torsion-free Abelian (and discrete by total disconnectedness). Since $H_i \cap Z_i$ is normal in H_i, it follows that Q is normalised by U. Thus we can form the subgroup $U \cdot Q$ of H_i, which is topologically locally finite and hence compact since U was maximal compact in H_i. This implies $Q \leqslant U$ and in particular we have $Q \leqslant U \cap Z_i =: V_i$.

Since V_i is normalised by U and centralised by N_i, it is a compact normal subgroup of H_i contained in $H_i \cap Z_i$. By definition, this implies that V_i is contained in Q. Thus $Q = V_i$ and the quotient $(H_i \cap Z_i)/V_i$ is thus torsion-free Abelian.

Notice that Z_i contains N_j for all $j \neq i$. Therefore $N_i.Z_i$ is dense in G and, since H_i is open, it follows that $H_i \cap (N_i.Z_i) = N_i.(H_i \cap Z_i)$ is dense in H_i. Therefore the Abelian group $(H_i \cap Z_i)/V_i$ maps densely to $H_i/N_i.V_i \cong U/U \cap (N_i.V_i) = U/(U \cap N_i).(U \cap Z_i)$. We deduce that the latter is Abelian, as claimed.

Our next claim is that $G/N_i.Z_i$ is Abelian. Indeed, we have $G = U.N_i.Z_i$ since $N_i.Z_i$ is dense. We deduce that $G/N_i.Z_i \cong U/U \cap (N_i.Z_i)$, which may be viewed as a quotient of the group $U/(U \cap N_i).(U \cap Z_i)$. The latter is known to be Abelian by (i), which confirms the present claim.

Suppose now that each N_i contains some maximal compact subgroup U_i. The above discussion shows that the derived group $[G, G]$ is contained in the intersection $N := \bigcap_{i=1}^p N_i.Z_i$. In other words the quotient G/N is Abelian, and it only remains to show that

$$N = N_1 \cdots N_p.\mathscr{Z}(G).$$

Let $g \in N_1.Z_1 \cap N_2.Z_2$ and write $g = n_1 z_1 = n_2 z_2$ with $n_i \in N_i$ and $z_i \in Z_i$. Then $n_1^{-1} z_2 = z_1 n_2^{-1}$ belongs to $Z_1 \cap Z_2$ since $N_i \subseteq Z_j$ for all $i \neq j$. Since $Z_1 \cap Z_2 = \mathscr{Z}_G(N_1.N_2)$, we deduce that $g \in N_1.N_2.\mathscr{Z}_G(N_1.N_2)$. This shows that $N_1.Z_1 \cap N_2.Z_2 \subseteq N_1.N_2.\mathscr{Z}_G(N_1.N_2)$. Since the opposite inclusion obviously holds true, we have in fact $N_1.Z_1 \cap N_2.Z_2 = N_1.N_2.\mathscr{Z}_G(N_1.N_2)$. A straightforward induction now shows that

$$\bigcap_{i=1}^{p} N_i.Z_i = N_1 \cdots N_p.\mathscr{Z}_G(N_1 \cdots N_p).$$

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Since $N_1 \cdots N_p$ is dense in G, we have $\mathscr{Z}_G(N_1 \cdots N_p) = \mathscr{Z}(G)$, from which the assertion (ii) follows. □

Quasi-products with Ad-closed quasi-factors. The following gives a simple criterion for a quasi-product to be direct.

LEMMA II·4. *Let G be a locally compact group that is a quasi-product with quasi-factors $N_1, \ldots, N_p$.*

If $N_1, \ldots, N_{p-1}$ are Ad*-closed and centrefree, then $G \cong N_1 \times \cdots \times N_p$.*

Proof. We work by induction on p, starting by noticing that the statement is empty for $p = 1$. Since $[N_1, N_i] \subseteq N_1 \bigcap N_i = 1$ for $i > 1$, we deduce that $N_2 \cdots N_p \subseteq \mathscr{Z}_G(N_1)$. In particular $N_1.\mathscr{Z}_G(N_1)$ is dense in G. From Lemma I·4 and the fact that N_1 has trivial centre, it follows that $G \cong N_1 \times \mathscr{Z}_G(N_1)$. By projecting G onto $\mathscr{Z}_G(N_1)$, we deduce that the product $N_2 \cdots N_p$ is dense in $\mathscr{Z}_G(N_1)$. Thus $\mathscr{Z}_G(N_1)$ is the quasi-product of $N_2, \ldots, N_p$. The desired result follows by induction. □

It will be shown in the next subsection that, conversely, a group N which is not Ad-closed may often be used to construct a non-trivial quasi-product having N as a quasi-factor, see Example II·8 below.

Non-direct quasi-products and dense analytic normal subgroups. We propose a general scheme to construct quasi-products out of a pair of topological groups M, N together with a faithful continuous M-action by automorphisms on N. The intuition is that M plays the role of *some* adjoint completion of N appearing in a quasi-direct product with two quasi-factors isomorphic to N. The precise set-up is as follows.

Let M, N be topological groups and $\alpha : M \hookrightarrow \mathrm{Aut}(N)$ an injective continuous representation. In complete generality, *continuity* shall mean that the map $M \times N \to N$ is jointly continuous; therefore, when considering locally compact groups, it suffices to assume that α is a continuous homomorphism for the Braconnier topology on $\mathrm{Aut}(N)$.

In order to formalise the idea that M is a generalisation of the Ad-closure $\overline{\mathrm{Ad}(N)}$, we assume throughout

$$\mathrm{Ad}(N) \subseteq \alpha(M) \qquad \text{and} \qquad \overline{\alpha^{-1}(\mathrm{Ad}(N))} = M.$$

Thus $\alpha(M)$ is indeed intermediate in $\mathrm{Ad}(N) \subseteq \alpha(M) \subseteq \overline{\mathrm{Ad}(N)}$. The trivial case, i.e. direct product, of our construction will be characterised by $\alpha(M) = \mathrm{Ad}(N)$. On the other hand, already $\alpha(M) = \overline{\mathrm{Ad}(N)}$ will produce interesting examples.

We denote by $\alpha_\Delta : M \hookrightarrow \mathrm{Aut}(N \times N)$ the diagonal action, which is still injective and continuous. We form the semi-direct product

$$H := (N \times N) \rtimes_{\alpha_\Delta} M,$$

which is a topological group for the multiplication

$$(n_1, n_2, m)(n_1', n_2', m') = (n_1\alpha(m)(n_1'), n_2\alpha(m)(n_2'), mm').$$

We observe that the set

$$Z := \left\{(n, n, m) : \alpha(m) = \mathrm{Ad}(n)^{-1}\right\}$$

is a subgroup of H and we write $G := H/Z$. For convenience, we write $N_1 = N \times 1$ and $N_2 = 1 \times N$, which we view as subgroups of H.

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PROPOSITION II·5.

(i) *Z is a closed normal subgroup of H (thus we consider G as a topological group).*

(ii) *The morphism $N_i \to G$ is a topological isomorphism onto its image, which is closed and normal in G; we thus identify N_i and its image. The resulting quotients G/N_i are topologically isomorphic to M.*

(iii) *The morphism $N \times N \to G$ has dense image; the latter is properly contained in G if and only if $\alpha(M) \neq \mathrm{Ad}(N)$. The kernel is the diagonal copy of $\mathscr{Z}(N)$ (in particular, if N is centre-free, G is a quasi-product).*

(iv) *$\mathscr{Z}_G(N_i) = N_{3-i}$; in particular, if N is centre-free, so is G.*

(v) *If N is topologically simple, then G is characteristically simple. Moreover, G cannot be written non-trivially as a direct product unless $\alpha(M) = \mathrm{Ad}(N)$.*

Proof. (i) The fact that Z is closed follows from the fact that the diagonal in $N \times N$ is closed and that α is continuous. A computation shows that $N \times N$ centralises Z, whilst M normalises it; hence Z is normal.

(ii) The morphism $N_1 \to G$ is continuous and injective. Suppose that a net $(n_\beta, 1, 1) \in N_1$ converges to some (n, n', m) modulo Z. Then there are nets $\nu_\beta \in N$, $\mu_\beta \in M$ with $\alpha(\mu_\beta) = \mathrm{Ad}(\nu_\beta)^{-1}$ such that $(n_\beta \nu_\beta, \nu_\beta, \mu_\beta)$ tends to (n, n', m). Thus n_β converges in N (to nn'^{-1}) and hence the morphism is indeed closed. The image is normal by definition and the case of N_2 is analogous.

It is straightforward to show that $H = N_i.Z.M$ and that $N_i.Z \cap M = 1$. Therefore $H/N_i.Z \cong M$, as claimed.

(iii) The density is equivalent to the density of $Z.(N \times N)$ in H, which follows from the conditions on α. The additional statement on the image follows from the canonical identification of coset sets

$$G/(N \times N) \;\cong\; M/\alpha^{-1}(\mathrm{Ad}(N)).$$

The description of the kernel is due to the fact that $(N \times N) \cap Z$ consists of those $(n, n, 1)$ with $\mathrm{Ad}(n) = 1$.

(iv) Suppose $(n_1, n_2, m) \in H$ commutes with N_1 modulo Z. Thus, for every $x \in N_1$ there is $(\nu, \nu, \mu) \in Z$ with

$$(n_1\alpha(m)(x), n_2, m) \;=\; (xn_1\alpha(m)(\nu), n_2\alpha(m)(\nu), m\mu).$$

The last two coordinates show that μ and ν are trivial. It follows that $\alpha(m)(x) = n_1^{-1}xn_1$. Thus $\alpha(m) = \mathrm{Ad}(n_1)^{-1}$ and hence (n_1, n_2, m) belongs to $N_2.Z$. The statement follows by symmetry and using the description of the kernel of $N \times N \to G$ obtained above.

(v) We can assume that N has trivial centre. Notice that the involutory automorphism of $N \times N$ defined by $(u, v) \mapsto (v, u)$ extends to a well defined automorphism of H which descends to an automorphism ζ of G swapping the two factors of the product $N_1.N_2$.

Let now $C < G$ be a (topologically) characteristic closed subgroup. Assume first that $C \cap N_1 = 1$. Then $1 = \zeta(C \cap N_1) = C \cap \zeta(N_1) = C \cap N_2$. Thus C centralises $N_1.N_2$ and is thus contained in $\mathscr{Z}(G)$ since $N_1.N_2$ is dense. Therefore we have $C = 1$ by (iv) in this case.

Assume now that $C \cap N_1 \neq 1$. Then N_1 is contained in C since N is topologically simple by hypothesis. Transforming by the involutory automorphism ζ shows that N_2, and hence also $N_1.N_2$, is then contained in C, which implies that $C = G$ since C is closed and $N_1.N_2$ is dense. Thus G is indeed characteristically simple, as desired.

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The above arguments show moreover that N_i are minimal closed normal subgroups of G. This implies that if G splits as a direct product $G \cong L_1 \times L_2$ of closed normal subgroups, then, upon renaming the factors, we have $N_i < L_i$ for $i = 1, 2$. It follows that $L_i < \mathscr{Z}_G(N_{3-i}) = N_i$ by (ii). Thus $N_i = L_i$. In view of (iii), this implies that G does not split non-trivially as a direct product of closed subgroups provided $\alpha(M) \neq \mathrm{Ad}(N)$. □

Our goal is now to present some concrete situations with M and N locally compact.

Example II·6. Let M, N be totally disconnected locally compact groups and let $\varphi : N \to M$ be a continuous injective homomorphism[2] whose image is dense and normal in M. In particular, the conjugation action of M on $\varphi(N)$ induces a homormorphism $\alpha : M \to \mathrm{Aut}(N)$; however α need not be continuous in general. However α is indeed continuous in the following cases:

(i) N is discrete and $\varphi(N) \leqslant \mathscr{QZ}(M)$;

(ii) $M = \overline{\mathrm{Ad}(N)}$ and $\varphi = \mathrm{Ad}$.

Of course these two cases are not mutually exclusive. One is then in a position to invoke Proposition II·5, which provides a totally disconnected locally compact group G that is a quasi-direct product with two copies of N as quasi-factors.

It is now easy to construct non-trivial quasi-products of totally disconnected groups by exhibiting a group N satisfying the required conditions.

Example II·7. Let N be one of the discrete groups described in Example I·10. This example yields a locally compact completion M and a continuous homomorphism $\alpha : M \to \mathrm{Aut}(N)$ such that $\mathrm{Ad}(N) \leqslant \alpha(M) \leqslant \overline{\mathrm{Ad}(N)}$. If N is simple, the group G provided by Proposition II·5 is characteristically simple. In this way, we obtain various examples of characteristically simple locally compact groups which are quasi-products but do not split as direct products. Notice however that in these examples N is not finitely generated and the corresponding G is never compactly generated.

Another way to satisfy the conditions of Example II·6 is to start with a group N which is not Ad-closed, but whose adjoint closure $M = \overline{\mathrm{Ad}(N)}$ is locally compact. We proceed to describe a concrete example. Part of the interest of the example is that N will be compactly generated, which implies that the associated groups M and G will be both compactly generated. Indeed, consider a compact generating set Σ for N and U any compact open subgroup of M. Then $U \cup \varphi(\Sigma)$ generates M, since $\langle U \cup \varphi(\Sigma)\rangle$ is open and contains a dense subgroup. Thus M is compactly generated, and so is H as well as all its quotients, including G.

Example II·8. Consider the semi-direct product

$$N = \mathrm{SL}_3\big(\mathbf{F}_p(\!(t)\!)\big) \rtimes \mathbf{Z},$$

where the cyclic group $\mathbf{Z}$ is any infinite cyclic subgroup of the Galois group $\mathrm{Aut}\big(\mathbf{F}_p(\!(t)\!)\big)$. Then N is not Ad-closed, but $\mathrm{Aut}(N)$ is locally compact, as follows from Theorem I·6. Moreover, the cyclic group $\mathbf{Z}$ normalises every characteristic subgroup of the compact open subgroup

$$\mathrm{SL}_3\big(\mathbf{F}_p[\![t]\!]\big) < \mathrm{SL}_3\big(\mathbf{F}_p(\!(t)\!)\big),$$

from which it easily follows that $\mathbf{Z}$ contains an unbounded asymptotically central sequence.

[2] Of course $\varphi(N)$ is only analytic when N is metrisable, but this is the standard situation to which the subsection title refers.

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Notice furthermore that N is centrefree. The easiest way to see this is by noticing that N acts minimally without fixed point at infinity on the Bruhat–Tits building associated with $\mathrm{SL}_3\big(\mathbf{F}_p(\!(t)\!)\big)$ (see [**CM09**, Theorem 1·10]). Thus Proposition II·5 may be applied. In conclusion, we find that the group

$$\frac{\Big(\big(\mathrm{SL}_3\big(\mathbf{F}_p(\!(t)\!)\big) \rtimes \mathbf{Z}\big) \times \big(\mathrm{SL}_3\big(\mathbf{F}_p(\!(t)\!)\big) \rtimes \mathbf{Z}\big)\Big) \rtimes \overline{\mathrm{Ad}\,\big(\mathrm{SL}_3\big(\mathbf{F}_p(\!(t)\!)\big) \rtimes \mathbf{Z}\big)}}{\Big\{(z, z, \mathrm{Ad}(z)^{-1}) : z \in \mathrm{SL}_3\big(\mathbf{F}_p(\!(t)\!)\big) \rtimes \mathbf{Z}\Big\}}$$

provides an example of a compactly generated totally disconnected locally compact group with trivial quasi-centre (in particular centrefree) which is a non-trivial quasi-product. However, this example is not characteristically simple.

Open problems. Corollary D naturally suggests the following question.

Question II·9. Is there a compactly generated characteristically simple locally compact group G that is a quasi-product with at least two simple quasi-factors, but which does not split non-trivially as a direct product?

Finally, we mention two related problems.

Question II·10. Is there a compactly generated topologically simple locally compact group G which contains a proper dense normal subgroup?

Question II·11. Is there a compactly generated topologically simple locally compact group G which is not Ad-closed?

As explained in Example II·8, a positive answer to the latter implies a positive answer to both Questions II·9 and II·10. Moreover, Proposition II·1 implies that a positive answer to Question II·9 also gives a positive answer to Question II·10.

On dense normal subgroups of topologically simple groups. We close this appendix with the following result, due to N. Nikolov [**Nik09**, Proposition 2], which provides however severe restrictions on possible non-Hausdorff quotients of topologically simple groups.

PROPOSITION II·12. *Let M be a compactly generated totally disconnected locally compact group which is locally finitely generated. Assume that M has no non-trivial compact normal subgroup. Then any dense normal subgroup contains the derived group $[M, M]$. In particular, if M is topologically simple, then it is abstractly simple if and only if it is abstractly perfect.*

Proof. See [**Nik09**, Proposition 2]. The statement from *loc. cit.* requires M to be topologically simple, but only the absence of *compact* normal subgroups is used in the proof. □

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REFERENCES

[**Are46**] R. ARENS. Topologies for homeomorphism groups. *Amer. J. Math.* **68** (1946), 593–610.
[**BEW08**] Y. BARNEA, M. ERSHOV and T. WEIGEL. Abstract commensurators of profinite groups. *Trans. AMS* (to appear).
[**Bir34**] G. BIRKHOFF. The topology of transformation-sets. *Ann. of Math.* (2) **35** (1934), no. 4, 861–875.

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[**BM00**] M. Burger and S. Mozes. Groups acting on trees: from local to global structure. *Inst. Hautes Études Sci. Publ. Math.* (2000), no. 92, 113–150 (2001).

[**BM02**] M. Burger and N. Monod. Continuous bounded cohomology and applications to rigidity theory. *Geom. Funct. Anal.* **12** (2002), no. 2, 219–280.

[**Bou60**] N. Bourbaki. *Éléments de mathématique. Première partie. (Fascicule III.) Livre III; Topologie générale. Chap. 3: Groupes topologiques. Chap. 4: Nombres réels*, Troisième édition revue et augmentée, Actualités Sci. Indust., No. 1143. (Hermann, 1960).

[**Bou71**] N. Bourbaki. *Éléments de mathématique. Topologie générale. Chapitres 1 à 4* (Hermann, 1971).

[**Bou74**] N. Bourbaki. *Éléments de mathématique. Topologie générale, chap. 5 à 10 (nouvelle édition)* (Hermann, 1974).

[**Bra48**] J. Braconnier. Sur les groupes topologiques localement compacts. *J. Math. Pures Appl.* (9) **27** (1948), 1–85.

[**Cap09**] P.-E. Caprace. Amenable groups and Hadamard spaces with a totally disconnected isometry group. *Comment. Math. Helv*. **84** (2009), 437–455.

[**CER08**] L. Carbone, M. Ershov and G. Ritter. Abstract simplicity of complete Kac-Moody groups over finite fields. *J. Pure Appl. Algebra* **212** (2008), no. 10, 2147–2162.

[**CH06**] P.-E. Caprace and F. Haglund. On geometric flats in the CAT(0) realization of Coxeter groups and Tits buildings. *Canad. J. Math.* **61** (2009), no. 4, 740–761.

[**CM09**] P.-E. Caprace and N. Monod. Isometry groups of non-positively curved spaces: structure theory. *J Topology* **2** (2009), no. 4, 661–700.

[**DdSMS99**] J. D. Dixon, M. P. F. du Sautoy, A. Mann and D. Segal. *Analytic Pro-p Groups*, second ed. Cambridge Studies in Advanced Math., vol. 61 (Cambridge University Press, 1999). MR MR1720368 (2000m:20039)

[**Dix96**] J. Dixmier. *Les Algèbres d'Opérateurs dans L'espace Hilbertien (Algèbres de von Neumann)*. Les Grands Classiques Gauthier-Villars (Éditions Jacques Gabay, 1996), Reprint of the second (1969) edition.

[**Dug66**] J. Dugundji. *Topology* (Allyn and Bacon Inc., 1966).

[**Fre36**] H. Freudenthal. Topologische Gruppen mit genügend vielen fastperiodischen Funktionen. *Ann. of Math.* (2) **37** (1936), no. 1, 57–77.

[**Gb46**] P. Gol′berg. The Silov II-groups of locally normal groups. *Rec. Math. [Mat. Sbornik] N.S.* **19(61)** (1946), 451–460.

[**GLS94**] D. Gorenstein, R. Lyons and R. Solomon. The classification of the finite simple groups, *Math. Surv. Monogr.* vol. 40 (American Mathematical Society, 1994).

[**GM67**] S. Grosser and M. Moskowitz. On central topological groups. *Trans. Amer. Math. Soc.* **127** (1967), 317–340.

[**GM71**] S. Grosser and M. Moskowitz. Compactness conditions in topological group. *J. Reine Angew. Math.* **246** (1971), 1–40.

[**GP57**] A. M. Gleason and R. S. Palais. On a class of transformation groups. *Amer. J. Math.* **79** (1957), 631–648.

[**Gui73**] Y. Guivarc'h. Croissance polynomiale et périodes des fonctions harmoniques. *Bull. Soc. Math. France* **101** (1973), 333–379.

[**HR79**] E. Hewitt and K. A. Ross. *Abstract harmonic analysis. Vol. I*, second ed. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 115 (Springer-Verlag, 1979). Structure of topological groups, integration theory, group representations.

[**Iwa49**] K. Iwasawa. On some types of topological groups. *Ann. of Math.* (2) **50** (1949), 507–558.

[**Kel75**] J. L. Kelley. *General Topology* (Springer-Verlag, 1975). Reprint of the 1955 edition [Van Nostrand, Toronto, Ont.], Graduate Texts in Math., No. 27.

[**KK44**] S. Kakutani and K. Kodaira. Über das Haarsche Mass in der lokal bikompakten Gruppe. *Proc. Imp. Acad. Tokyo* **20** (1944), 444–450.

[**LW70**] D. H. Lee and T.-S. Wu. On CA topological groups. *Duke Math. J.* **37** (1970), 515–521.

[**Mar91**] G. A. Margulis. Discrete subgroups of semisimple Lie groups. *Ergeb. Math. Grenzgeb* (3), vol. 17 (1991).

[**Men65**] J. L. Mennicke. Finite factor groups of the unimodular group. *Ann. of Math.* (2) **81** (1965), 31–37.

[**Mon01**] N. Monod. *Continuous bounded cohomology of locally compact groups*. Lecture Notes in Math. vol. 1758 (Springer-Verlag, 2001).

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[Moz98] S. MOZES. *Products of trees, lattices and simple groups*. Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), no. Extra Vol. II (1998), pp. 571–582 (electronic). MR MR1648106 (2000a:20056).

[MŚ59] M. MACBEATH and S. ŚWIERCZKOWSKI. On the set of generators of a subgroup. *Nederl. Akad. Wetensch. Proc. Ser. A* 62 = *Indag. Math*. **21** (1959), 280–281.

[MvN43] F. J. MURRAY and J. VON NEUMANN. On rings of operators, IV. *Ann. of Math*. (2) **44** (1943), 716–808.

[MZ55] D. MONTGOMERY and L. ZIPPIN. Topological Transformation Groups (Interscience Publishers, 1955).

[Nik09] N. NIKOLOV. Strange images of profinite groups. Preprint, 2009.

[Pat88] A. L. T. PATERSON. Amenability. Math. Surv. Monogr., vol. 29 (1988).

[Pla65] V. P. PLATONOV. Lokal projective nilpotent radicals in topological groups. *Dokl. Akad. Nauk BSSR* **9** (1965), 573–577.

[Pla77] P. PLAUMANN. Automorphismengruppen diskreter Gruppen als topologische Gruppen. *Arch. Math. (Basel)* **29** (1977), no. 1, 32–33.

[Pon39] L. S. PONTRJAGIN. *Topological Groups*. Princeton Mathematical Series, v. 2 (Princeton University Press, 1939). Translated from the Russian by Emma Lehmer.

[Sch25] O. SCHREIER. Abstrakte kontinuierliche Gruppen. *Abh. Math. Hamburg* **4** (1925), 15–32.

[Tit64] J. TITS. Algebraic and abstract simple groups. *Ann. of Math*. (2) **80** (1964), 313–329.

[TV99] S. THOMAS and B. VELICKOVIC. On the complexity of the isomorphism relation for finitely generated groups. *J. Algebra* **217** (1999), no. 1, 352–373.

[Uša63] V. I. UŠAKOV. Topological $\overline{FC}$-groups. *Sibirsk. Mat*. Ž. **4** (1963), 1162–1174.

[vD30] D. VAN DANTZIG. Ueber topologisch homogene Kontinua. *Fund. Math*. **15** (1930), 102–125.

[vD31] D. VAN DANTZIG. Studien over topologische algebra (proefschrift). Ph.D. thesis (Groningen, 1931).

[vDvdW28] D. VAN DANTZIG and B. L. VAN DER WAERDEN. über metrisch homogene Räume. *Abh. Math. Hamburg* **6** (1928), no. 2, 367–376.

[Vie27] L. VIETORIS. Über den höheren Zusammenhang kompakter Räume und eine Klasse von zusammenhangstreuen Abbildungen. *Math. Ann*. **97** (1927), 454–472.

[Wan69] S. P. WANG. The automorphism group of a locally compact group. *Duke Math. J.* **36** (1969), 277–282.

[Wei40] A. WEIL. L'intégration dans les groupes topologiques et ses applications. *Actual. Sci. Ind.* no. 869 (Hermann et Cie., 1940).

[Wil71] J. S. WILSON. Groups with every proper quotient finite. *Proc. Camb. Philos. Soc*. **69** (1971), 373–391.

[Wil94] G. A. WILLIS. The structure of totally disconnected, locally compact groups. *Math. Ann.* **300** (1994), no. 2, 341–363.

[Wil07] G. A. WILLIS. Compact open subgroups in simple totally disconnected groups. *J. Algebra* **312** (2007), no. 1, 405–417.

[Wu71] T.-S. WU. On (CA) topological groups. II. *Duke Math. J.* **38** (1971), 513–519.

[Zer76] D. ZERLING. (CA) topological groups. *Proc. Amer. Math. Soc*. **54** (1976), 345–351.

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